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Quantum Topological Geometrodynamics

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The description of 3-space as a spacelike 3-surface X of the space $H = M^4 \times CP_2$ (Product of Minkowski space and two-dimensional complex projective space CP_2) and the idea that particles correspond to 3-surfaces of finite size in H are the basic ingredients of topological geometrodynamics (TGD), an attempt at a geometry-based unification of the fundamental interactions. The observations that the Schrödinger equation can be derived from a variational principle and that the existence of a unitary S-matrix follows from the phase symmetry of this action lead to the idea that quantum TGD should be derivable from a quadratic phase-symmetric variational principle for some kind of superfield (describing both fermions and bosons) in the configuration space consisting of the spacelike 3-surfaces of H. This idea as such has not led to a calculable theory. The reason is the wrong realization of the general coordinate invariance. The crucial observation is that the space Map(X, H), the space of maps from an abstract 3-manifold X to H, inherits a coset space structure from H and can be given a Kahler geometry invariant under the local $M^4 \times SU(3)$ and under the group Diff of X diffeomorphisms. The space Map(X, H) is taken as a basic geometric object and general coordinate invariance is realized by requiring that superfields defined in Map(X, H) are diffeo-invariant, so that they can be regarded as fields in Map(X, H)/Diff, the space of surfaces with given manifold topology. Super-d'Alembert equations are found to reduce to a simple algebraic condition due to the constant curvature and Kähler properties of Map(X, H). The construction of physical states leads by local $M^4 \times SU(3)$ invariance to a formalism closely resembling the quantization of strings. The pointlike limit of the theory is discussed. Finally, a formal expression for the S-matrix of the theory is derived and general properties of the S-matrix are discussed.

1. INTRODUCTION

Topological geometrodynamics (TGD) is a geometry-based attempt to unify the fundamental interactions based on the idea that classical space-time can be regarded as a submanifold of some higher dimensional space H (Pitkänen, 1981, 1983, 1985, 1986). Once the postulate about representability as a submanifold is accepted, one is led rather naturally to the following scenario.

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1. The concepts of particle and 3-space generalize and, in a certain sense, are unified. Particles (in a very general sense of the word) are identified as spacelike 3-surfaces of H, so that a topological classification of particles and particle reactions emerges. Classical 3-space with particles is identified as a topologically trivial 3-surface to which the particlelike 3-surfaces are "glued."

2. The natural requirement that isometries of the space H are symmetries of the theory leads to the identification of the space H as the Cartesian Product $M^4 \times CP_2$ of Minkowski space and of the space CP_2, the complex projective space (Eguchi et al., 1980; Gibbons and Pope, 1978). The isometry group of the space CP_2 is identified as the color group. Thus, one can identify color gravitational interactions as interactions coupling to the isometry charges of the space H.

3. The so-called induction procedure allows one to define gauge potentials on the submanifolds of H as field quantities induced from the spinor connection of the space H. It turns out that these gauge potentials can be identified as electroweak gauge potentials.

4. The geometrization of spectroscopy is achieved. The choice explains the quantum numbers associated with a single particle family and the family replication phenomenon has a natural topological explanation. One can imagine several dynamical scenarios in which either leptons or quarks or both appear as elementary fermions.

Concerning the form of the quantum dynamics based on this general framework, perhaps the most important achievement hitherto is the realization that the formulation of the theory, whatever it may be, should be free of arbitrary physically relevant parameters. This means that all dimensional parameters (gravitational coupling, masses, etc.) should be related to the length scale of CP_2 by some predictable dimensionless numbers (scale invariance is broken by the curvature of CP_2). Also, dimensionless couplings should be predictions of the theory.

The use of the conventional quantization methods to construct a quantum theory which "predicts everything" is highly questionable, since these methods typically describe interactions as nonlinearities in the action defining the theory, so that various coupling constants are arbitrary parameters at least at the classical level.

Thus, the quantization philosophy adopted in Pitkänen (1986) was, in concise form, "Do not quantize!" The new line of thought was based on the following observations:

1. The ordinary Schrödinger equation is obtained from a variational principle and the associated action is quadratic with respect to the probability amplitude.

2. The existence of a Hilbert space scalar product and of a unitary S-matrix results from the conservation law of probability, which in turn

follows from the phase symmetry of the quadratic action.

From these observations we abstracted (Pitkänen, 1986) our basic recipe of a quantum theory:

1. Assume the existence of a configuration space SH endowed with metric structure (spacelike submanifolds of H).

2. Postulate a variational principle for the probability amplitudes defined in SH with the property that the associated action is quadratic with respect to the probability amplitudes and invariant under phase symmetries.

The action principle is fixed to high degree by requiring that:

3. The probability amplitudes are geometric objects and the action is defined by a Lagrangian density invariant under the coordinate transformations of the space SH. More concretely, the quantum equations of motion should correspond to "massless" (no free physically relevant parameters) d'Alembert-type equations in configuration space endowed with a metric.

Furthermore, it is natural to assume that:

4. The geometry of the configuration space (spacelike 3-submanifolds of the space H) is induced from the geometry of H (metric, Riemannian connection, and spinor structure).

Since the state functionals must be capable of describing both bosons and fermions and states of arbitrarily high fermion number, it is natural to postulate:

5. A state functional is a Grassmann algebra-valued "scalar superfield" (Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983; Berezin, 1966). The generators of the configuration space (3-submanifold of H) are in one-to-one correspondence with the configuration space spinors.

In order to understand how this program might lead to a unitary S-matrix, consider the following argument. Configuration space SH is obtained by gluing together spaces $SH(t, n)$ corresponding to 3-manifolds with a given number of components ("particle number") with given topologies t. The points common to $SH(t1, m)$ and $SH(t2, m)$ correspond to surfaces topologically intermediate between manifold topologies $t1$ and $t2$ and are singular as manifolds.

Consider now a state functional corresponding to a well-defined particle number n and thus restricted to $SH(n)$. The uncertainty principle in SH (!) implies that this state cannot be stationary, but begins to disperse to other parts of SH with different particle numbers. Clearly, this process leads to occurrence of particle reactions.

We have already made an attempt to formulate a mathematical theory based on these general ideas (Pitkänen, 1986). In this paper we give a formulation which differs in one important respect from the previous brute force approach: the general coordinate invariance is realized differently.

In the previous formulation (Pitkänen, 1986) we derived expressions for the line element of the metric and related quantities of the configuration space and obtained rather a complicated-looking formulation. The global geometry of the configuration space was not discussed.

The present formulation is based on the observation that single-particle configuration space can be regarded as the space Map(X, H)/Diff, where G is the space of maps from X (with given topology) to $H = M^4 \times CP_2$ and Diff is the group of diffeomorphisms of X.

The space Map(X, H)/Diff as such is not very simple geometrically, but Map(X, H) is. The point is that it can be regarded as a coset space G/F, where G is the group Map(X, $M^4 \times SU(3)$) and F is the group Map(X, $SU(2) \times U(1)$). The group operation is given by the group operation of $M^4 \times SU(3)$ taken pointwise in X.

This space can be made a constant-curvature space with G-invariant geometry and thus the local $M^4 \times SU(3)$ acts as its isometry group. The calculation of various geometric quantities is expected to reduce to a mere group-theoretic task.

Thus the following approach suggests itself. Instead of Map(X, H)/Diff, use the space Map(X, H) as the basic object and realize the general coordinate invariance by posing the following requirements: (1) The allowed superfields in G/F are diffeoinvariant and can be regarded as superfields in Map(X, H)/Diff; (2) the G-invariant geometry of G/F is also diffeoinvariant.

This approach has some far-reaching implications. The symmetry group of the theory is G and its Lie algebra is a Kac–Moody-type algebra. Thus, the quantum mechanical states must form irreducible representations of this group and we have good hopes of understanding the general features of the physical state space by group-theoretic considerations alone. The same also applies to the evaluation of the S-matrix elements.

The plan of the paper is as follows:

In Section 2 I discuss the problem of defining SH as a manifold and consider in detail the geometrization of the space G/F in a G- and diffeoinvariant manner. I find that the requirement of diffeoinvariance implies Kähler (and thus symplectic) structure in this space and that the metric is quite unique.

The construction of the Kähler metric leads in a natural manner to a central extension of the Lie-algebra of G and it is found that this extension is very similar to that encountered in the string model. The so-called quantization can be understood purely geometrically. It corresponds to the introduction of a Kähler potential term into the ordinary covariant derivative in G/F.

The construction of spinor structure is also considered and a representation of gamma matrices in the space of H-spinors defined on X is given.

In Section 3 I construct the superfield formalism and in the case of constant-curvature space derive the general solution of super-d'Alembert equations.

Section 4 is devoted to the symmetries of the theory. The realization of the color symmetry is discussed in detail. It is shown that positive norm requirement fixes the form of the super-d'Alembertian uniquely and implies matter–antimatter asymmetry and a generalized chiral invariance (separately conserved baryon and lepton numbers) and its spontaneous breaking. $N = 1$ supersymmetry at the pointlike limit is also a consequence of this symmetry.

The general features of the Lie-algebraic state construction in the infinite-dimensional case are discussed and it is found that the resulting scenario resembles very closely that encountered in the string model, in that physical states can be regarded as diffeoinvariant representations of the centrally extended Kac–Moody algebra associated with the local $M^4 \times SU(3)$.

Section 5 is devoted to the discussion of the pointlike limit of the theory. The general rules for the construction of states are formulated and it is shown that the experimentally observed particles can be identified from the spectrum.

Section 6 is devoted to the construction of the S-matrix. The so-called bare states are defined as state functionals restricted to a subset of the configuration space corresponding to fixed manifold topology t. Stationary states are defined as continuations of the bare state functionals to state functionals in the whole configuration space. A formal solution of the continuity conditions is derived and conditions guaranteeing the uniqueness of the continuation are deduced. Finally, an explicit expression for S-matrix as a unitary matrix transforming bare states to stationary state is derived.

The pointlike limit of the theory is discussed briefly and it is shown how one can understand the coupling strengths and selection rules from the form of the general solution of the super-d'Alembertian.

2. ABOUT THE STRUCTURE OF THE CONFIGURATION SPACE

In this section I shall first discuss the general structure of the configuration space and then perform the diffeoinvariant geometrization of the space $\mathrm{Map}(X, H)$.

2.1. The General Structure of the Configuration Space

In earlier work (Pitkänen, 1981, 1983, 1985, 1986) I formulated the theory in the space SH of surfaces of the space H. The space of surfaces was defined in the following manner:

1. Allowed 3-surfaces are spacelike and can have arbitrarily many disjoint components. Thus, *SH* divides into subsets with definite component number. The connected surfaces are clearly fundamental geometric objects.

2. A surface was defined as a subset of *H*, which is locally manifold. Thus surface can have self-intersections and pinches, etc. In earlier publications (Pitkänen, 1981, 1983, 1985) I have discussed the description of the particles as submanifolds of *H*.

3. The singular submanifolds of *H*, which correspond to topology, changing transitions of 3-manifolds, play a central role in the description of the interactions in TGD. These surfaces are intermediate between two manifold topologies.

A rough classification of the various topology changes and a description of various particle reactions in terms of the intermediate topologies was suggested in earlier papers (Pitkänen, 1981, 1983, 1985). It is natural to require that all allowed singular surfaces have topologies intermediate between two manifold topologies.

The set *SH* of surfaces was defined as the union of the sets $SH(t_i)$ and the sets $SH(t_{ij})$, where t_{ij} is a manifold with topology intermediate between the topologies t_i and t'_j:

$$SH = \bigcup_i SH(t_i) \cup [\bigcup_{ij} SH(t_{ij})]$$

In this union the manifolds with intermediate topologies were understood as limiting cases of nonsingular manifolds.

It was argued that the subsets of *SH* consisting of submanifolds of *SH* with a fixed topology are of the same dimension as *SH* itself and that the subsets consisting of singular manifolds in some sense form a "measure-zero" subset of *SH*.

The idea that submanifolds with a fixed topology form a subset of *SH* having the same dimension as *SH* derives from their property of being open sets of *SH*. That these sets are open follows from the invariance of the property of being submanifold with a given topology under small deformations of the surface.

The subsets of *SH* consisting of surfaces that are singular as submanifolds of *H* are not open, since an arbitrary small deformation can lead to a final state that corresponds to manifold topology (perform a small deformation near the "reaction vertex").

Since the singular manifolds are in certain sense limiting cases of regular manifolds, the sets $SH(t_{ij})$ for a given topology t_i must belong to the compactification of the set $SH(t_i)$.

For reasons explained in the introduction, I take as the basic geometric object in the present approach, not the space $SH(t) = \text{Map}(t, H)/\text{Diff}$, but the mapping space Map(t, H). This means that configuration space must be replaced by the union of the corresponding mapping spaces:

$$\bigcup_i \mathrm{Map}(t_i, H) \cup \left[\bigcup_{ij} \mathrm{Map}(t_{ij}, H)\right]$$

The mapping spaces should be glued together by identifying the mapping spaces Map(t_{ij}, H) as limiting cases of the spaces Map(t_i, H), $i = 1, 2$.

One can indeed find an elegant description for the limit $t_1 \to t_{12} \leftarrow t_2$, and the geometrization of the space Map(t_{ij}, H) does not differ from the geometrization of Map(t, H) in any essential respect. It is not clear whether one could regard the spaces Map(t_{ij}, H) as measure-zero sets of the whole mapping space. Fortunately, this question does not seem to be relevant for the formulation of the theory.

The superiority of the mapping space approach manifests itself in the calculation of the S-matrix elements. In the previous approach the evaluation of the S-matrix elements necessitates the calculation of overlap integrals of superfields over some submanifold of SH consisting of submanifolds with topologies intermediate between initial and final topologies. Thus, one faces the problem of evaluating the integration measure (formally defined as the measure associated with the induced submanifold metric).

In the present approach the calculation of the overlap integrals needed to obtain S-matrix elements can be performed using the same formalism as used for the calculation of the various scalar products in Map(t, H). One can perform these integrals in the nice geometry of the mapping space Map(t_{ij}, H) and it is to be expected that overlap integrals are fixed to a high degree by symmetry considerations alone [local $M^4 \times SU(3)$ is the isometry group of Map(t_{ij}, H)].

2.2. Geometrization of Map(X, H)

2.2.1. General Considerations

The configuration space $SH(t)$ associated with a single particle of fixed topology t can be related to the space Map(X, H) of maps from an abstract 3-manifold X to the space H. The image $f(X)$ of X defines a 3-surface in H. The maps g and f, which differ only by a diffeomorphism d of X; $g = f \circ d$, obviously define the same surface.

The most straightforward approach to the construction of the quantum theory in the space $SH(t)$ of surfaces obtained by identifying the diffeo-related points of Map(X, H) (t denotes the topology of X)

$$SH(t) = \mathrm{Map}(X, H)/\mathrm{Diff}$$

This approach (Pitkänen, 1986) is perhaps not the most elegant formulation, since the geometrization of the space $SH(t)$ is not straightforward.

The new approach is based on the observation that any diffeoinvariant field defined in Map(X, H) can be regarded as a field in $SH(t)$. Thus, the alternative recipe for the realization of the general coordinate invariance is as follows.

Take the space Map(X, H) as the basic geometric object and give it diffeoinvariant geometric structure and require that the physically acceptable solutions of the field equations are diffeoinvariant.

The main reason behind the change of our attitude (Pitkänen, 1986) is that the space Map(X, H) in the case of $H = M^4 \times CP_2$ can be regarded as a coset space (Freed, 1985; de la Harpe, 1972),

$$\mathrm{Map}(X, H) = G/F$$

$$G = \mathrm{Map}(X, M^4 \times SU(3))$$

$$F = \mathrm{Map}(X, SU(2) \times U(1))$$

G is the loop space consisting of maps $X \to M^4 \times SU(3)$ and has natural gauge group-like structure defined by the pointwise multiplication of maps,

$$f_1 \circ f_2 = (m_1 + m_2, s_1 \circ s_2) \tag{1}$$

The space F is the loop space of maps $X \to SU(2) \times U(1)$ and can be regarded as a subgroup of the group G.

The gauge group $SU(2) \times U(1)$ is present simply because the identification of the points of loop space differing by a mere local $SU(2) \times U(1)$ transformation implies that the map can be regarded as a map to $M^4 \times CP_2$ as required.

Of course, the group operation does not necessarily lead to a spacelike 3-surface, and thus the causality requirement implies that the allowed maps correspond to an open subset of the loop group. I believe that this restriction is not essential.

What makes this space so promising with regard to the formulation of the theory is that it can be regarded as an infinite-dimensional symmetric space (de la Harpe, 1972; Helgason, 1962; Freed; 1985). The following properties of the symmetric spaces makes clear why the space G seems to be the correct object with regard to the formulation of the theory.

1. The isometry group of the coset space G/F is G in the natural left invariant metric of G/F. In our case it is the local $M^4 \times SU(3)$ and can be identified as the local gauge group of color gravitational forces. Requiring that the various metric structures are diffeoinvariant, the symmetry group of the theory extends to the semidirect product of Diff(X) and G.

2. *Since the metric of* G/F is obtained by left translation from the metric in some fixed point of G/F, it is clear that the Riemannian connection, the vielbein connection, and the curvature tensor are describable in purely Lie-algebraic terms. Symmetric spaces are constant-curvature spaces; the curvature tensor is a covariantly constant quantity, invariant under G.

3. The Lie algebra associated with the loop space G is a Kac–Moody algebra (Lie algebra of local gauge transformations) and since it is the isometry group of the configuration space, we can expect that the solutions of the super-d'Alembertian can be classified into irreducible unitary diffeoinvariant representations of this group. Also, the evaluation of the S-matrix elements and interaction vertices is facilitated by the enormous symmetries of the theory.

Of course, the formulation based on the use of the space $SH(t)$ might be equally promising, if one could give for the space $SH(t)$ a G-invariant metric. I have, however, not been able to show whether this space allows a G-invariant metric or not. The difficulty is basically that Diff is not subgroup of G and thus its relation to the group G is different than the relation of the group F.

2.2.2. Geometry of Map(X, H)

In order to understand the general structure of the loop space, it is useful to study the structure of the loop space G (Freed, 1985; de la Harpe, 1972).

The tangent space of G corresponds to the Lie algebra LG of local $M^4 \times SU(3)$ gauge transformations. This Lie algebra is generated by the elements

$$J_m^A = s_m \times g^A \tag{2}$$

where the functions s_m define a complete scalar function basis S for the surface X and g^A are generators of $M^4 \times SU(3)$. Thus, the Kac–Moody algebra is direct product of the function algebra S and the finite-dimensional Lie algebra g,

$$LG = S \otimes g \tag{3}$$

It is natural to divide this algebra into two parts,

$$LG = g \oplus S_1 \otimes g = g \oplus t \tag{4}$$

where g corresponds to constant scalar functions and generates symmetries in "center-of-mass" degrees of freedom. S_1 is the space of nonconstant scalar functions.

A natural scalar product in g is defined by the direct sum of the Minkowski metric and the canonical scalar product of simple Lie algebras,

$$(X, Y) = \mathrm{Tr}(\mathrm{Ad}\, X, \mathrm{Ad}\, Y) \tag{5}$$

Ad X is the representation matrix of the $SU(3)$ generator X in the adjoint representation.

A diffeoinvariant scalar product in the space of scalar functions is defined by the integral over X in the integration measure dx defined by the induced metric defined by the imbedding (Pitkänen, 1981, 1983, 1985, 1986)

$$(f, g) = \int fg\, dx \tag{6}$$

The scalar product in $S_1 \times g$ is defined by the tensor product of the metrics associated with g and S_1. Of course, the diffeoinvariance of the scalar product at the Lie algebra level does not yet guarantee its invariance at the group level. In fact, this scalar product as such does not provide a diffeo-invariant scalar product in G.

In the case of the coset space CP_2 we can assume that only the Lie algebra generators corresponding to the orthogonal complement of the $H = SU(2) \times U(1)$ subalgebra appear in the expansion of the tangent vector of the factor space and the scalar product is defined by the unique Ad H-invariant metric.

The left invariant metric in the coset space G/F can be constructed by mimicking the corresponding construction for the finite-dimensional spaces (Freed, 1985; de la Harpe, 1972; Helgason; 1962).

1. Assume that the Ad F-invariant metric is given at some arbitrarily chosen point g of G. The Ad F is the Jacobian of any transformation

$$g \to fgf^{-1}; \qquad f \in F \tag{7}$$

at the point g. Forgetting for a moment the requirement of diffeoinvariance, one could construct the metric of the space G/F mimicking directly the finite-dimensional construction.

2. Define the metric at other points of the coset space by the left translation h; the metric at point hg is obtained by contracting its indices with the Jacobian $Dh(g)$ of the map $g \to hg$.

How can the obtained metric be made diffeoinvariant at all points of G/F? In order to be diffeoinvariant, the scalar function part of the metric should be expressible in a form (6) at all points of G/F. The simplest scalar product (6), however, depends on the determinant of the induced metric and this depends on the point of G/F.

Clearly, one should somehow modify the scalar product for scalar functions. In fact, there are several possibilities, since one can insert suitable differential operators into the simplest definition of the scalar product.

We shall find in the sequel that the construction of diffeo- and G-invariant metrics is closely related to the problem of finding nontrivial central extensions for the algebra LG; central extension defines a metric in Map(t, H). Diffeoinvariance in fact implies that the metric is a Kähler metric. Therefore, we shall treat the two problems on the same footing.

2.3. Central Extension

2.3.1. General Considerations

The Lie algebra of the isometry group differs from the Lie algebra appearing in the quantized string model in one important respect: all M^4-type generators commute. In the quantized string model we have an algebra of annihilation and creation operators and this feature seems to be crucial for the physical interpretation of the theory.

This result is not fatal to our quantization program, however. I shall show that "quantization" has a purely geometric interpretation in the TGD approach.

The point is that in the string model the algebra generating physical states can be regarded as a central extension of the commuting algebra, which would correspond to the commuting infinite-dimensional isometry algebra. This means the addition of a term proportional to the unit matrix to the commutation relations of the "classical" commuting Lie algebra, and its effect is the same as that of the quantization.

That this central extension has a purely geometric description becomes clear from the following finite-dimensional example. In order to get a respectable spinor structure in CP_2 one must add a suitable multiple of a CP_2 Kähler potential to the ordinary spinor connection defined by the vielbein.

Remarkably, the addition of the Kähler potential can also be regarded as a central extension of the original $SU(3)$ Lie algebra. This modification is obtained by replacing in the differential operator representation of the Lie-algebra generators the ordinary derivatives with $U(1)$ covariant derivatives defined by the Kähler potential.

The modified commutation relations contain a term which is the contraction of the curvature form of the Kähler form with the Lie-algebra generators appearing in the commutator. This modification indeed defines central extension; the commutation relations are given by

$$[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} + nJ(X, Y)Id \tag{8}$$

where J is the Kähler form of CP_2 and Id is the unit matrix.

Jacobi identities are satisfied because the curvature form is closed as a two-form. This condition is equivalent to the requirement that the allowed central extensions of the Lie algebra are in one-one correspondence with the nontrivial cocycles of the second cohomology group of the space where the group in question acts.

Since this procedure introduces central extension purely geometrically, it is reasonable to try to generalize it to the infinite-dimensional case.

1. Construct a Kähler structure in the space t spanned by nonconstant Lie-algebra generators. The corresponding Kähler form must be covariantly constant and thus also closed. The existence of Kähler structure implies that the part t of the tangent space of G/F can be complexified:

$$t = t_+ + t_- \tag{9}$$

The complexification corresponds to the use of a complex function basis. t_- is the complex conjugate of t_+.

The Kahler form has nonvanishing elements only between t_+ and t_- and the elements of the Kähler form are, apart from sign factors, equal to the elements of the metric tensor.

This is an important result concerning the diffeoinvariance of the central extension. If one can find a left G-invariant and diffeoinvariant Kähler form, then one has also obtained a metric with these properties and the problem of finding a left- and diffeoinvariant metric for G/F is solved.

We shall find that the left invariance of the Kähler form must be understood in the generalized sense described earlier besides performing a local gauge rotation for the Lie-algebra generators, left translation also induces a transformation in the Hilbert space S_1 of nonconstant scalar functions in X.

2. Define the central extension by replacing the ordinary derivatives in the differential operator representation of the G-algebra with the covariant derivatives defined by the Kähler potential.

2.3.2. General Form of the Extension

The first task is that of finding the general form of the central extension at the Lie-algebra level. Let the central extension be given by

$$[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} + f(X, Y) Id \tag{10}$$

(Id denotes the unit matrix). Here one can use the following results (Goddard-Olive, 1983; Kac, 1983) from the construction of central extensions in the string model.

1. The Jacobi identities imply the condition

$$\begin{aligned} &f(X, Y, Z) + f(Y, Z, X) + f(Z, X, Y) \\ &\quad + \partial_X f(Y, Z) + \partial_Y f(Z, X) + \partial_Z f(X, Y) = 0 \end{aligned} \tag{11}$$

defining what is meant by a 2-cocycle in group cohomology (Kac, 1983). Here the derivatives $\partial_X f$ are Lie derivatives of the function f under the infinitesimal transformation defined by X.

2. Derivations of the Lie algebra defined by the condition

$$d[X, Y]=[dX, Y]+[X, dY] \tag{12}$$

of the Lie algebra give rise to possible central extensions via the formula

$$f(X, Y)=k(dX, Y) \tag{13}$$

Here $(\cdot, \cdot)$ is some scalar product for the Lie-algebra elements. Now we can use the natural scalar product defined by the induced metric in X.

Cocycle conditions are satisfied provided the terms of the type $\partial_X f(Y, Z)$ vanish identically in the cocycle conditions, which means that derivation is left invariant.

3. Any element A of the finite Lie-algebra element defines a trivial derivation through the formula

$$dX=[A, X] \tag{14}$$

As already mentioned, in the case of CP_2 this kind of extension leads to a nontrivial result; one obtains a respectable spinor structure and, depending on the nature of the extension $t=0$ or $t=1$, representations of the color group carrying anomalous hypercharge. Also, fractionation of the electromagnetic charge becomes possible.

The commutator action of the color hypercharge is equivalent to the contraction of the corresponding vector fields with Kähler form, which indeed corresponds to a nontrivial element in the second cohomology group of CP_2.

It turns out that the requirements of diffeo- and left invariance under G allow this central extension only for the rigid part g (constant scalar functions of the G-Lie algebra).

4. Any element of the Lie algebra of Diff(X) defines a derivation

$$f(X, Y)=(j\cdot\nabla X, Y) \tag{15}$$

Here j denotes some vector field defined in X.

Assuming G-invariance (I shall discuss this requirement later), the cocycle condition reduces to the condition satisfied by the scalar functions f, g, and h:

$$(j\cdot\nabla(fg), h)+\text{cyclic perm}=0 \tag{16}$$

The cocycle condition is satisfied provided the diffeomorphism generator has vanishing divergence in the induced metric,

$$\nabla\cdot j=0 \tag{17}$$

Thus, the vector field j defines central extension if it generates a volume-preserving diffeomorphism in the induced metric. Since j is an identically conserved current in the induced metric, it must be determined by the embedding.

The requirement that central extension (and metric and Kähler form) is diffeoinvariant is satisfied provided its dependence on the induced metric somehow cancels. This is achieved if j is some topological current that does not depend on the metric. In the case of $M^4 \times CP_2$ the projection of the Kähler form indeed defines this kind of current,

$$j = * J \tag{18}$$

Here $*$ denotes the Hodge $*$-operator (contraction of the Kähler form with three-dimensional permutation symbol). The dependence of the scalar product on the metric cancels, since this current is inversely proportional to the square root of the metric determinant.

This current has vanishing divergence for purely topological reasons (Bianchi identities). Note that this term is present only in the three-dimensional case and the conserved charge associated with this current is the homology equivalence class of the two surface in question [the Kähler form can be regarded as the generator of $H_2(CP_2)$].

The metric determined by this extension is not Hermitian unless one adds a suitable boundary term to the extension. The necessity of this term is easy to understand by transforming the central extension term to a form where the operator $j \cdot \nabla$ acts on the function on the right-hand side of the scalar product.

The Hermiticity is achieved provided one associates the extension term defined by the boundary vector field given by

$$i = -2e * B(g_3/g_2)^{1/2} \tag{19}$$

to each boundary component. Here B denotes the Kähler potential and $*$ is the two-dimensional Hodge $*$-operator. The sign factor e ensures that the orientation of the boundary component can be regarded as that induced from X.

5. The central extension defined by j is expected to lead to a strongly degenerate metric; for all zero-eigenvalue solutions the diagonal element of the metric vanishes. A natural attempt to get rid of this degeneracy is to look for some boundary term restricted to the zero-eigenvalue subspace of j.

Because of their two-dimensionality, the boundary components carry a natural Kähler structure uniquely defined by the induced metric. In real coordinates the Kähler form is simply the two-dimensional permutation symbol $\varepsilon^{\alpha\beta}$.

Hence the cocycle defined by the formula

$$f(X, Y) = (*(dPX \wedge dPY))\, dx \tag{20}$$

(P projects to the space of zero eigenvalues of $j \cdot \nabla$) does not depend on the induced metric and is a good candidate for central extension.

This part of the central extension does not depend on the imbedding of X and thus the requirements of G- and diffeoinvariance are automatically satisfied.

Summarizing, the general form of the central extension is given by

$$f(X \times g^A, Y \times g^B) = g(X, Y)(g^A, g^B) + k_2 \operatorname{Tr}(\mathrm{Ad}_y P_0 g^A, P_0 g^B)$$

$$g(X, Y) = k \int_X j \cdot \nabla XY\, dX - 2k \int_{\delta X} i \cdot \nabla XY\, dX$$

$$+ k_1 \int_{\delta X} *(dPX \wedge dPY)\, dx \tag{21}$$

Here P_0 is the projector to the subspace of constant scalar functions; P projects to the subspace of zero-eigenvalue functions of $j \cdot \nabla$; and y denotes the color hypercharge generator.

2.3.3. G-invariance of the Central Extension

The G-invariance of the central extension is not a trivial issue, since the concept of left G-action is somewhat problematic. The simplest definition of left G-action is the following one.

1. G acts only on the finite-dimensional Lie-algebra generators via left translation and leaves the scalar function basis invariant. For the scalar function basis associated with the boundary part of the central extension one can apply this definition. The reason is that central extension does not depend on the imbedding.

The interior part of the central extension, however, depends on imbedding, since the operator $j \cdot \nabla$ depends on the induced Kähler form. Thus, the matrix elements of the central extension change in the left translation. As a consequence, the additional terms in the cocycle condition are non-vanishing.

A straightforward calculation shows that the G-invariance condition applied to the infinitesimal color rotation

$$\delta s^k = s_m(x) j^{Ak}$$

reduces to the form

$$\int * (dH_A \wedge ds_m \wedge ds_n) s_r\, dx = 0; \qquad s_i \in S, \quad i = m, n, r \tag{22}$$

Here H_A denotes the Hamiltonian of the color isometry generated by the Lie-algebra generator $j^{Ak} \leftrightarrow g^A$.

This condition is not satisfied unless one poses some additional constraints on the scalar functions appearing in (22). An obvious condition of this kind is the requirement that allowed scalar functions depend only on a single coordinate variable of X locally.

This coordinate variable is naturally the coordinate variable changing along the field lines of the magnetic field defined by the Kähler form.

This kind of condition makes the metric strongly degenerate; 3-manifolds become effectively one-dimensional as far as the properties of the interior part of the central extension are concerned. One can regard this kind of degeneracy as an undesirable feature.

One can, however, invent a more general definition of G-action and G-invariance:

2. G also acts on the scalar function basis. A natural definition of this action is obtained by requiring that the left invariant scalar function basis be an eigenfunction basis of the operator $j \cdot \nabla$ and possibly some differential operators commuting with it. This requirement is expected to define left translation of the scalar function basis uniquely for sufficiently small left translations. If the diagonal elements of the operator $j \cdot \nabla$ are left invariant in the diagonal basis, the central extension is left invariant.

The invariance of these matrix elements is not at all a trivial property. In the sequel we shall find, however, that under small deformations of X the diagonal matrix elements of the operator $j \cdot \nabla$ are indeed invariant. Invariance follows from the fact that these elements are interpretable as topological invariants. Thus, the central extension and thus also the geometry of Map(X, H) are left invariant in this generalized sense.

2.3.4. Kähler Structure Associated with the Central Extension

In the finite-dimensional case the central extension defined the Kähler potential leads to a central extension term proportional to the Kähler form. Clearly, the matrix defined by the matrix elements of the operator $j \cdot \nabla$ can be identified as an integer multiple of the Kähler form (and appropriate part of the metric) associated with the central extension.

The Kähler structure makes possible the construction of a complexified basis for the Lie algebra. This operation corresponds in the finite-dimensional case to the complexification of the $SU(3)$ Lie algebra by requiring that the Lie algebra forms an eigenbasis of the Kähler potential. The result is the representation of the algebra in the standard eigenbasis of hypercharge Y and isospin I_3.

In the infinite-dimensional case the complexification corresponds to the requirement that the function basis S_1 is an eigenbasis of the differential operator $j \cdot \nabla$ and possibly some other differential operators commuting with it (analogues of isospin generator).

Since j has a vanishing divergence, it is an anti-hermitian operator and the eigenfunctions are orthogonalizable. For the topological current the metric disappears from the scalar product and we can require the scalar functions to be eigenfunctions of the operator $g^{1/2}j\cdot\nabla = D$,

$$g^{1/2}j\cdot\nabla s = ips \tag{23}$$

rather than the operator $j\cdot\nabla$ as in the general case. Thus, the dependence of the eigenfunction basis on the metric disappears.

The eigenfunction basis is in general complex and the complex conjugate eigenfunctions correspond to opposite eigenvalues. The Lie-algebra generator and its complex conjugate correspond to a single pair of creation and annihilation operators. The product of two elements in the basis is also an eigenfunction of D and eigenvalues are additive in multiplication.

The central extension defined by the Kähler form is invariant under diffeomorphisms of X and also under the infinite parameter group of canonical transformations of CP_2 leaving the induced Kähler form invariant. In fact the canonical transformations act like $U(1)$ gauge transformations on the Kähler potential.

This means that the spectrum of the operator D is same for two 3-surfaces having CP_2 projections related by a canonical transformation. The dependence of M^4 coordinates on X coordinates is irrelevant.

For the standard Kac–Moody Lie algebra the matrix elements of the Kähler form appearing in the central extension are integers. This property plays an essential role in the construction of the unitary representations for the Kac–Moody algebra. We shall now show that diagonal elements of the interior part of the central extension are indeed integer-valued provided the complexified functions with nonvanishing j eigenvalue can be regarded as maps from X to S^1,

$$s = \exp(i\theta), \qquad \theta\in R \tag{24}$$

This restriction does not imply any loss of generality; in ordinary Fourier analysis the same assumption is made.

The essential observation is that the map $f: X\to H$ can be extended to a map $g: X\to H\times S^1$ by the rule $x\to(h,\theta)$. These maps can be classified by the third cohomology equivalence class of the image surface $g(X)$. This equivalence class is given by the product

$$w_3 = w_1\wedge w_2 \tag{25}$$

where w_1 is the first cohomology class of S^1 and w_2 is the second cohomology class of CP_2. The expression of this homology equivalence class for $f(X)$ is given by the projection of the form w_3 and is equal to the diagonal element of D.

One can understand the general features of the eigenvalue equation associated with j rather easily. Let the surface X possess a region where the induced Kähler form is nonvanishing. Assume also that this region has three-dimensional CP_2 projection as it generically has locally (the assumption about two-dimensionality of the projection does not alter the argument essentially). The projection of the surface in CP_2 is representable as a surface for which some Hamiltonian H is constant.

By a suitable choice of canonical coordinates (P, Q, P_1, Q_1) one can represent the surface in the form

$$P = H = \text{const}$$

The coordinates of X can be chosen so that the imbedding is expressible in the form

$$(Q, P_1, Q_1) = (x_1, x_2, x_3)$$

The nonvanishing component of the induced Kähler form J is constant.

In these coordinates the eigenvalue equations reduce to the following form:

$$\partial s/\partial Q = ips \tag{26}$$

The equation is thus of the same form for all surfaces with nonvanishing Kähler form. A general solution to this equation is of the form

$$\exp(ipQ)\exp[f(P_1, Q_1)] \tag{27}$$

Since the solutions obey the multiplicative superposition property, p must be an integer multiple of some constant,

$$p = na \tag{28}$$

The operators $\partial/\partial P_1$ and $\partial/\partial Q_1$ define two vector fields commuting with the vector field j, and their eigenvalues are integer multiples of some basic eigenvalue. Thus, the scalar function basis can be regarded as a generalization of the plane wave basis.

In the plane wave basis labeled by the integers $(n_1, n_2, n_3) = \bar{n}$, the modified commutation relations reduce to the very simple form

$$[J^A_{\bar{m}}, J^B_{\bar{n}}] = f^{AB}_C J^C_{\bar{m}+\bar{n}} + kn\delta(\bar{n} + \bar{m})Id + \text{boundary term} \tag{29}$$

The integer n is the third homology equivalence class defined earlier. As far as the interior part of the central extension is concerned, the algebra has the structure of a centrally extended Kac–Moody algebra appearing in the string model.

Also, the diffeomorphism generators can be chosen to be an eigenbasis of the generators $\partial/\partial x_i$. One can construct the basis of diffeogenerators from the mutually commuting generators $\partial/\partial x_i$ by forming a direct product of this basis with the complexified scalar function basis.

As a consequence, one can associate with each diffeomorphism generator a positive or negative eigenvalue as in the case of the string model. In the construction of the representations of the centrally extended Lie algebra of local $M^4 \times SU(3)$ the possibility to classify diffeogenerators in this manner is of central importance.

How uniquely is the constant appearing in the interior part of the central extension determined? From the representation theory of centrally extended Kac–Moody algebras (Kac, 1983) we know that it is an integer multiple of some uniquely determined quantity. Thus we obtain the same result as in the case of CP_2.

I shall now show that the diagonalized form of the boundary part of the central extension is integer-valued. Denote by z the local complex coordinate defined on δX. Thus one can represent the space t of boundary restrictions of the zero-eigenvalue scalar functions as a direct sum $t_+ + t_-$, where t_+ and t_- correspond to holomorphic and antiholomorphic functions, respectively.

One can interpret the map $(z, \bar{z}) \to (s, \bar{s})$ as a section in the complexified tangent bundle $T_c\, \delta X$. The homology equivalence class of the image of δX in $T_c\, \delta X$ is integer-valued, since $T_c\, \delta X$ has nontrivial second homology group. The homology equivalence class of the image of δX, the winding number of the δX vector field defined by $(s, \bar{s})$, is proportional to the quantity

$$\int [\partial(s, \bar{s})/\partial(z, \bar{z})]\, dz \wedge d\bar{z} \tag{30}$$

Thus the diagonal elements of the extension are integer-valued. The nondiagonal matrix elements between maps with different winding numbers are expected to cancel. In sphere topology the orthogonality is evident: the maps are proportional to the powers of z and the orthogonality is obvious. Thus the boundary part of the central extension is indeed integer-valued, as it should be.

2.3.5. Topological Structure of the Central Extension

The following observations suggest that the topological structure of the central extension plays a central role in the understanding of the general features of the theory.

1. The magnetic field defined by the Kähler form is divergenceless and has the same topological structure as the velocity field associated with an incompressible three-dimensional flow.

2. The field lines of the magnetic Kähler field are closed lines and define what is called a Seifert–Threlfall fibration of the 3-manifold (Jehle,

1977; Seifert and Threlfall, 1931, 1932, 1950; Rolfsen, 1976). The simplest fibrations are closed vortices, which correspond to magnetic fields confined inside a torus, and one can describe this magnetic field topologically by two winding numbers n_1 and n_2, which indicate how many times the closed field line winds around the central axis and the circular axis.

Clearly, the vector field defining the interior extension takes a derivative along the magnetic field line and one can associate with each elementary magnetic structure of this kind a complete basis of scalar functions.

3. The three-dimensionality of the 3-space leads to an additional richness in the topological structure: the vortices can become knotted and linked (Rolfsen, 1976).

4. The presence of boundary components brings in additional structure; nonclosed magnetic flux loops connecting two boundary components become possible.

5. There are some quite intriguing points of contact with superconductivity (Goldstein, 1975). The scalar function corresponds to the order parameter of the superconducting phase (phase of the probability amplitude describing Cooper pairs). The diagonal element of central extension can be regarded as an expectation value of the magnetic flux in the "state" defined by the scalar function and its integer-valuedness corresponds to the quantization of magnetic flux in the sense in which it is defined in superconductivity.

The formation of Cooper pairs in superconductivity might be interpreted as a change of the magnetic structure of X; magnetic flux tubes are formed between the boundary components corresponding to electrons. In a similar manner the penetration of a magnetic field into superconductors of type II in the form of vortices can be interpreted as a change of the magnetic structure.

6. The constant-curvature property of the metric does not seem to be completely global; rather, $\mathrm{Map}(X, H)$ divides into constant-curvature pieces characterized by the topology of the induced magnetic field.

The worst possible degeneracy of the metric of $\mathrm{Map}(X, H)$ occurs for surfaces for which the induced Kähler form vanishes. The induced Kähler form vanishes when the surface in question has CP_2 projection, which is a Lagrangian submanifold (Pitkänen, 1981, 1983, 1985, 1986) defined by the property that the projection of the Kähler form to it vanishes. Lagrangian manifolds are in general two-dimensional submanifolds of CP_2; they are analogous to the submanifold $P = \text{const}$ of the classical phase space with (q, p) coordinates. Certainly, this set of 3-surfaces has measure zero, since a small deformation leads to a surface with a nonvanishing Kähler field.

For these surfaces the central extension is given by the boundary part of this term. For closed 3-surfaces of this type the degeneracy is even greater.

This degeneracy phenomenon gives support to the idea that elementary particles in certain circumstances correspond to boundary components of the 3-manifold.

7. Magnetic structure is reflected in the properties of the scalar function basis and thus also in the properties of the metric and spinor structure of the configuration space. For example, spinor components are in direct correspondence with the scalar function basis.

These observations suggest that magnetic structure should be regarded as a carrier of the information needed to arrange individual elementary particles to various forms of macroscopic matter. Indeed, the topology of the magnetic fields allows unlimited possibilities to store information. Therefore, the concept of magnetic structure might prove to be useful in the description of certain many-particle phenomena. For example, the concepts used to describe the gross features of the properties of fluid flow (eddiness, degree of turbulence) are expected to be useful in the description of magnetic structure and thus also in the description of some features of the many-particle system with given magnetic structure.

2.4. Calculation of the Various Geometric Quantities

The G-invariance of the geometry makes it possible to evaluate explicit expressions for the various quantities related to the geometry of the configuration space. One can calculate the quantity in question at suitably chosen point of G/F and use left translation to obtain its value at other points of G/F.

The metric is the direct sum of three terms: the H metric, a part corresponding to the interior part of the central extension, and a part corresponding to the boundary part of the central extension,

$$H = h \oplus H_I \oplus H_\delta \tag{31}$$

The interior part of the metric tensor separates into a tensor product of the H metric and the metric in the space of the scalar functions. In the complexified eigenfunction basis labeled by eigenvalues one has

$$\begin{aligned} H(J^A_m, J^B_n) &= G(m, n) \times h(t^A, t^B) \\ G(\bar{m}, \bar{n}) &= N(\bar{m})\ \delta(\bar{m} + \bar{n}) \end{aligned} \tag{32}$$

$N(\bar{m})$ is the homology equivalence class for the image of X under the map defined by s_m.

The corresponding decomposition of the vielbein is given by

$$E^{\bar{A}} = [N(m)]^{1/2} G^{\bar{m}} \times e^A; \qquad \bar{A} = (m, A) \tag{33}$$

where we have defined the "reduced vielbein" $G^{\bar{m}}$ via the formula

$$G^{\bar{m}}_n = [(i/2)]^{1/2}[\delta(\bar{m} + \bar{n}) - i\,\delta(\bar{m} - \bar{n})] \tag{34}$$

The boundary part of the metric has the analogous decomposition

$$H(J^A_m, J^B_n) = H(m, n) + h(t^A, t^B)$$
$$H(m, n) = N(m)\,\delta(m+n) \tag{35}$$

$N(m)$ is the winding number associated with the eigenfunction s_m.

The corresponding decomposition of the vielbein is given by

$$E^{\bar{A}} = N^{1/2} G^m \times e^A; \qquad \bar{A} = (m, A) \tag{36}$$

The calculation of the connection and curvature tensor is not quite so simple, since the metric is not necessarily nondegenerate. In the left invariant vector field basis J^A_m one can use the following formula (Freed, 1985) for the calculation of the Riemann connection:

$$Y = (\mathrm{Ad}_x Y - \mathrm{Ad}^*_X Y - \mathrm{Ad}_Y X)/2 \tag{37}$$

Here Ad^*_X is the adjoint of the map Ad_X with respect to the scalar product defining the metric. For Ad_X one can derive an explicit representation directly from its definition (Freed, 1985):

$$(\mathrm{Ad}^*_X Y, Z) = (Y, \mathrm{Ad}_X Z) \tag{38}$$

Thus one obtains for the operator Ad^*_X the expression

$$Ad^*_X = -D^{-1} P \mathrm{Ad}_X D P$$
$$-2 \int dz\, \mathrm{Ad}_X P_0\, \partial/\partial z - 2 \int d\bar{z}\, \mathrm{Ad}_X P_0\, \partial/\partial \bar{z} \tag{39}$$

Here the operator D is the differential operator $g^{1/2} j \cdot \nabla$; z denotes the complex coordinate for the boundary component, and P_0 and P project to the subspace of scalar functions with zero and nonzero eigenvalue of D, respectively.

The curvature tensor can be calculated from the general expression

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \tag{40}$$

In the eigenfunction basis the operator D can be replaced with the appropriate eigenvalue.

The general element of the curvature tensor is given by the expression

$$R(X, Y, Z, U) = A([X, [Y, Z]], U) + B([Y[Z, X]], U) \tag{41}$$

Here $R(X, Y, Z, U)$ denotes the contraction of the curvature tensor with the tangent vectors of configuration space interpreted as elements of the Lie algebra. A and B depend on the generators X, Y, Z, and U.

The Riemann tensor is proportional to the metric tensor. We expect the proportionality constant to be infinite. The curvature scalar is certainly infinite. These quantities do not appear in the super-d'Alembertian, although the curvature scalar appears in the square of the Dirac operator.

2.5. Spinor Structure in Map(X, H)

In previous work (Pitkänen, 1986) I suggested a definition of a configuration space spinor as a map that associates with a surface X an H-spinor field defined in X. One can, however, define configuration space spinors without any reference to the H-spinor field defined in X and the resulting formalism is more elegant (Stinespring, 1965).

As in the finite-dimensional case, we define gamma matrices by requiring that they generate the metric tensor H (Eguchi *et al.*, 1980)

$$\{\Gamma_{\bar{A}}, \Gamma_{\bar{B}}\} = 2H_{\bar{A}\bar{B}} Id \tag{42}$$

The indices $\bar{A}$ correspond to the pairs (n, A), where n refers to the scalar function basis and A refers to the H-vielbein.

We divide the gamma matrix algebra of Map(X, H) into the direct sum $g \oplus t$. The subspace g of gamma matrices corresponds to constant scalar functions and can be identified as the space of the H-gamma matrices. The subspace t of gamma matrices corresponds to nonconstant scalar functions. The gamma matrices in g and t are represented in the form

$$\begin{aligned} \hat{\Gamma}_A &= \Gamma_A \otimes Id; \qquad \hat{\Gamma}_A \in g \\ \hat{\Gamma}_{\bar{A}} &= \Gamma_9 \otimes \Gamma_{\bar{A}}; \qquad \hat{\Gamma}_A \in t \end{aligned} \tag{43}$$

respectively. The spinors of Map(X, H) are thus expressible as tensor products of H-spinors and the spinors associated with t.

The Kähler structure of the Map(X, H) makes it possible to transform the gamma matrices in t to an algebra of anticommuting annihilation and creation operators (see Appendix). The complexified gamma matrices are defined by the formula

$$\Gamma_{\pm}^{\bar{A}} = \Gamma^{\bar{A}} \pm iJ_{\bar{B}}^{\bar{A}} \Gamma^{\bar{B}} \tag{44}$$

and obey an algebra that reduces to that obeyed by annihilation and creation operators by multiplication with factors of $\sqrt{1}/2$,

$$\{\Gamma_{+}^{\bar{A}}, \Gamma_{-}^{\bar{B}}\} = 2H^{\bar{A}\bar{B}} Id; \qquad \{\Gamma_{\pm}^{\bar{A}}, \Gamma_{\pm}^{\bar{B}}\} = 0 \tag{45}$$

It is natural to represent complexified gamma matrices in terms of annihilation and creation operators, and configuration space spinors form a representation of the algebra of creation and annihilation operators defined by the bare gamma matrices.

The abstract Hilbert space where the annihilation and creation operators act is assumed to be same for all surfaces X. The universality of the spinor space is necessary, since in the calculation of S-matrix elements one must be able to relate the spinors associated with different 3-topologies to each other.

A given spinor in t is obtained by applying a finite number of creation-operator-type gamma matrices to a "vacuum spinor" annihilated by all annihilation-type gamma matrices ("vacuum spinor" is analogous to right-handed neutrino),

$$u = \prod_n a_n U_0, \qquad a_n U_0 = 0 \quad \forall n \tag{46}$$

This representation of spinors implies a considerable simplification concerning the calculational properties of the theory. The point is that only the annihilation-type gamma matrices appear in the super-d'Alembertian. Thus, in the calculation of scalar products and various quantities of physical interest, only a finite number of the gamma matrices appearing in the field equations give a nonzero result when applied to a given configuration space spinor.[2]

The spinors obtained by applying a single creation operator to the vacuum are in one-to-one correspondence with the spinor field basis of X. One can interpret these spinors as a counterpart of the Fourier components of the spinor field in conventional field theories.

The spinor components corresponding to the application of several creation operators to the vacuum have no counterpart in conventional field theories. It might well be that they imply an additional degeneracy analogous to the family replication phenomenon. Thus, the hope of obtaining a theory predicting three or four or even a finite number of fermion families seems to be rather meager!

Observe that for X with trivial magnetic structure one can associate the configuration space spinors with boundary components, since it is possible to choose the corresponding part of the eigenfunction basis so that each scalar function is concentrated around a single boundary component. Matter resides on boundaries for these surfaces.

One can construct a spinor connection in terms of the vielbein connection V using the standard defining formulas expressing the covariant constancy of the vielbein. The covariant derivative of a spinor field can be defined exactly as in the finite-dimensional case (Eguchi et al., 1980).

[2]Obviously the Fourier components of the so called Ramond field appearing in string models have interpretation as the complexified gamma matrices of the configuration space associated with string.

Experience with the finite-dimensional case suggests strongly that in the case of $H = M^4 \times CP_2$ the Kähler Potential term must be included in the definition of the spinor connection in order to obtain a well-defined spinor structure in the configuration space.

Thus, we expect that the spinor covariant derivative is given by

$$D = \partial + V + nBId \tag{47}$$

where B is the Kähler potential associated with the central extension of the G-Lie algebra; n is an odd integer.

The extension of the theory to the case of 3-manifolds with several components is quite straightforward. The scalar function basis can be written in the form ($X = X_1 \cup \cdots \cup X_n$)

$$g + t_1 + \cdots + t_n \tag{48}$$

Observe that center-of-mass degrees of freedom are not additive. The imbedding of the gamma matrices associated with t must span the whole Hilbert space. One can choose the imbedding so that the gamma matrices of X_i correspond to the following subalgebra of creation and annihilation operators spanned by

$$a_n, a_n^+, \qquad n = i + nk, \quad k \in N \tag{49}$$

3. SUPERFIELD FORMULATION

In this section I formulate the concepts of superfield and super-d'Alembertian. I show that under certain conditions, super-d'Alembert equations can be solved in closed local form.

3.1. Superfield and Super-d'Alembertian

The generalization of the concept of complex-valued probability amplitude to a Grassmann algebra-valued probability amplitude, superfield, offers an attractive possibility to describe both bosonic and fermionic states with arbitrary high fermion number using a single fieldlike quantity. I shall first develop the finite-dimensional formulation. The generalization to the infinite-dimensional case is trivial.

The key concept of the finite-dimensional superfield formulation is the local Grassmann algebra (Berezin, 1966) spanned by the "theta parameters", which are in one-to-one correspondence with the spinor basis associated with a given point of H; the superfield is simply a map associating with each point of H an element in this algebra. In the formulation of the superfield dynamics the concept of a "super covariant derivative," changing the fermion number by one unit, and that of "super-d'Alembertian" as a generalization of the ordinary d'Alembertian, play key roles.

Consider first the definition of the Grassmann algebra structure. Let $\{u_m\}$ be a complete orthonormalized basis of spinors at the point h of H and $\{\bar{u}_m\}$ the conjugate basis. Associate with each element u_m ($\bar{u}_m$) an anticommuting theta parameter $\theta_m(\bar{\theta}_m)$. The requirement that the quantities $\bar{u}\theta$ and $\bar{\theta}u$ are invariant under vielbein rotations implies that theta parameters transform as spinors under vielbein rotations.

Theta parameters generate a Grassmann algebra at the point h. By "globalizing" this concept, one is led to the concept of the "spinorial Grassmann algebra bundle" having as its fiber the local Grassmann algebra generated by the theta parameters. The spinorial Grassmann algebra bundle might be regarded as a spin-half version of the Grassmann algebra bundle generated by 1-forms of H.

Any element of the Grassmann algebra can be expressed as a polynomial of the theta parameters: in the obvious short-hand notation

$$S = S(M, N)\theta^M\bar{\theta}^N \tag{50}$$

The coefficients of the various monomials are complex numbers and behave as multispinors under vielbein rotations, since the Grassmann algebra elements must be invariant under vielbein rotations.

The superfield can be defined as map that associates to each point of the space H an element of the Grassmann algebra associated with that point. In a more advanced formulation, a superfield is a section in the Grassmann algebra bundle. The component representation of the superfield is obtained from the polynomial representation of the Grassmann algebra element.

An important feature differentiating between the concept of the superfield used in the supersymmetric field theories and in the present context is that now the components of the superfield are assumed to be complex numbers; in supersymmetric field theories the odd components are assumed to be anticommuting numbers (Stelle, 1983).

The conjugate $\bar{S}$ of the superfield is obtained by performing Dirac conjugation for the component multispinors and making the replacement $\theta \leftrightarrow \bar{\theta}$.

The scalar product for two superfields S and R is given by the formula

$$(S, R) = \int \bar{S}R\, D\theta\, D\bar{\theta}\, dh \tag{51}$$

where the integration over the Grassmann algebra is defined according to the usual rules (Berezin, 1966) and dh denotes the standard metric integration measure in H.

Super covariant derivatives are defined as

$$D_m^{\pm} = i^{1/2}[\partial\theta_m^{\pm} + (\bar{\theta}^{\mp}\Gamma)_m D_k] \tag{52a}$$

$$\bar{D}_m^{\pm} = i^{1/2}[\partial\bar{\theta}_m^{\pm} + (\Gamma^k\theta^{\mp})_m D_k] \tag{52b}$$

Here plus and minus denote the chiralities of the spinors. The derivative D_k is the usual covariant derivative containing a spinor connection part and a metric connection part. Theta parameters are by definition covariantly constant with respect to the covariant derivative D_k and the components of the superfield transform as multispinors.

In conventional theories (Stelle, 1983) and also in my previous work (Pitkänen, 1986) super covariant derivatives are not defined as here. Usually the term containing the covariant derivatives D_k is proportional to the imaginary unit and the factor $i^{1/2}$ is absent. The presence of the $i^{1/2}$ factor guarantees that the anticommutation relations of the modified super covariant derivatives are identical with those of the conventional theories, as one can easily verify.

The reason for modifying the definition of the super covariant derivative is the need to obtain a nontrivial scalar product for the solutions of the super-d'Alembertian. It turns out that this requirement excludes the conventional definition of the super covariant derivatives.

One can replace the gamma matrices appearing in the definition of the super covariant derivative with annihilation operator-type modified gamma matrices (see the Appendix), which can be defined for the configuration space having Kähler structure. This replacement leads in the infinite-dimensional case to decisive calculational simplifications.

This is due to the fact that configuration space spinors correspond to states created by a finite number of creation-type gamma matrices applied to a vacuum spinor annihilated by annihilation operator like gamma matrices. Therefore, only a finite number of modified gamma matrices appearing in the super covariant derivative give a nonvanishing result when applied to a given configuration space spinor.

It will be found that the presence of the modified gamma matrices makes it possible to reduce the field equations to a simple algebraic condition.

The general form of the super-d'Alambertian studied in this paper is given by the expression

$$\boxtimes = a\{\bar{D}_m^-, D_m^+\} + b\{\bar{D}_m^+, D_m^-\} \tag{53}$$

The variational principle given by

$$S = (S, \boxtimes S) \tag{54}$$

leads to the expected form of the super-d'Alembert equations.

We shall find that the requirement of positive-definite scalar product in the space of solutions implies that one of the coefficients a and b appearing in (53) must vanish.

The formal generalization of the finite-dimensional formulation looks quite straightforward; all formulas generalize as such to the infinite-dimensional case. Also, the generalization of the formalism to the case of nonconnected 3-manifolds is straightforward.

In practice it is enough to consider the connected case only. The reason is that for 3-manifolds with several components one can represent the solutions of the superfield as superpositions of the product of single-particle superfields.

3.2. General Solution of the Field Equations

3.2.1. Solution Ansatz

The constant-curvature and Kähler properties of the configuration space imply that field equations can be reduced to a simple algebraic condition for the super-d'Alembertian obtained by replacing ordinary gamma matrices by the annihilation operator-type modified gamma matrices (see the Appendix).

The solution ansatz is obtained as a generalization of the explicit solution to the so-called chiral condition (Stelle, 1983) and is given by

$$S = \exp(X)T; \qquad X = \bar{\theta}\nabla\!\!\!/\,\theta \tag{55}$$

Here $\nabla\!\!\!/$ is the (possibly modified) Dirac operator.

The solution ansatz can in certain sense be regarded as a superfield S obtained by replacing coordinate variables h^k appearing in the superfield T with the new coordinate variables

$$Y^k = h^k + \bar{\theta}\Gamma^k\theta$$

This replacement is defined by the Taylor expansion defined by the exponential term. The variables Y^k are constant with respect to the sum of the super covariant derivatives. If the configuration space is flat, one can indeed use the variables Y^k instead of h^k as coordinate variables and the field equations are expected to reduce to the condition

$$\partial\bar{\partial}T = 0 \tag{56}$$

The equation is purely algebraic and easily solvable. When configuration space is nonflat, the curvature form of the spinor connection is expected to change the situation somewhat.

In order to see what happens, one must commute the super-d'Alembertian with the exponential term, i.e., calculate the quantity

$$U^{-1}\boxtimes U; \qquad U = \exp(X) \tag{57}$$

or equivalently evaluate how the quantity

$$A = U\,\partial\bar{\partial}\,U^{-1} \tag{58}$$

differs from the operator ⊠.

By expanding the quantity in question in terms of the multiple commutators one obtains the following representation:

$$A = A_0 + \sum_n A_n/n!; \qquad A_n = [X, A_{n-1}]; \qquad A_0 = \partial\bar{\partial} \tag{59a}$$

The coefficients A_n are given by

$$\begin{aligned} A_1 &= \partial\theta_m^+(\hat{\Gamma}^k\theta^+)_m D_k + \partial\bar{\theta}_m^-(\bar{\theta}^-\Gamma^k)_m D_k \\ A_2 &= 2\hat{\Box}\,\bar{\theta}^-\theta^+; \qquad \hat{\Box} = \hat{H}^{kl}D_kD_l \\ A_3 &= \hat{H}^{kl}(D_mF_{nk} + 2F_{mk}D_n)\bar{\theta}^-\theta^+\bar{\theta}\hat{\Gamma}^k\theta \end{aligned} \tag{59b}$$

The terms A_n, $n > 4$, contain covariant derivatives of the curvature form of the spinor connection. For constant-curvature spaces all terms A_n, $n > 4$, vanish. If A_n vanishes for some n, then all A_m, $m > n$, vanish also.

The modified metric tensor appearing in the formulas (59a) and (59b) is defined through the anticommutator of the modified gamma matrices,

$$\hat{H}^{kl} = \{\hat{\Gamma}^k, \hat{\Gamma}^l\}/2 \tag{60}$$

When the Dirac operator appearing in the field equation is modified so that it contains only annihilation operator-type gamma matrices, this tensor is equal to the metric tensor of Minkowski spacd and thus its contraction with the curvature tensor of the configuration space vanishes identically. Thus, the field equations indeed reduce to the desired simple algebraic condition and we can solve them exactly! The basic reason for this drastic simplification is the Kähler structure of the space Map(X, H) implied in turn by the Kähler structure of CP_2.

So far the treatment has been completely general. The solution must, however, be such that it allows positive-definite norm. The study of the conserved probability current shows that a positive-definite scalar product can be generated dynamically provided the solution is proportional to the following term:

$$U = \exp(i\bar{\theta}^+K\theta^-) \tag{61}$$

where the matrix K connects only the theta parameters of the second chirality. This is so because in the scalar product the conjugation operation performed on the second member of the scalar product completes the expression so that all theta parameters are present.

If the super-d'Alembertian contains only the derivatives $\partial\theta^+$ and $\partial\bar{\theta}^-$, the exponential term is annihilated in the operator appearing in the condition

(56). Thus, the need to obtain a positive-definite scalar product necessitates the matter–antimatter asymmetric form of the super-d'Alembertian. Note, however, that the asymmetry is not present in the general form of the solution.

The dependence of the solution ansatz on the coordinate variables of the configuration space is arbitrary and it is clear that the solution obtained does not necessarily have conserved norm. Also, the convergence of the solution is by no means clear. The same phenomenon of course occurs for ordinary eigenvalue problems. What are the necessary conditions for obtaining a conserved scalar product?

In the finite-dimensional case the nonconservation of the scalar product is easy to understand. The solution has arbitrary space-time dependence and thus the scalar product is certainly time dependent. Clearly, one needs some kind of mass shell condition. The requirement that the solution satisfies the modified d'Alembert equation and has positive energy,

$$\hat{\Box} S = 0 \tag{62a}$$

$$P_0 \geq 0 \tag{62b}$$

is perhaps the most natural mass shell condition, but is by no means dictated by the field equations.

In finite-dimensional theory this kind of condition forces the solutions to be massless and the orthogonalization of different solutions is possible. The same thing happens in fact in the infinite-dimensional case (the modified d'Alembert operator is equal to the M^4 d'Alembertian). Thus, this condition can be only a good approximation to the truth (recall the natural mass scale given by the Planck mass).

A more general condition guaranteeing a conserved scalar product is based on the requirement that the superfields T generating the solution belong to unitary representations of some central extension of the local $M^4 \times SU(3)$. States inside the irreducible representation should be eigenstates of the operator

$$p^2 - kC \tag{63a}$$

where p^2 is the mass squared operator, C is the Casimir operator associated with "vibrational" degrees of freedom (nonconstant scalar functions), and k is some constant. The field equations pose no constraints on the value of this parameter. Neither does the requirement of irreducibility.

The value of k determines only the mass scales inside a representation. The requirements of convergence and single-valuedness for the superfield are expected to pose conditions on the possible values of k. The naive value for this constant is $k = 1$. The elementary particle mass scale is obtained if k is of the order of 10^{-38}.

The simplest solution ansatz is based on the assumption that the matrix K is proportional to the chiral projector P_-. In the finite-dimensional case this is certainly a natural choice and turns out to be a unique choice in the infinite-dimensional case. The underlying reason is that the calculation of the scalar product in the infinite case involves integration over an infinite number of theta parameters; when K is equal to the projection operator the calculations simplify decisively and lead to well-defined finite results. For a general matrix K one would encounter the difficulties associated with the definition of the trace and determinant of an infinite-dimensional matrix.

It should be noticed that the solution ansatz for the field equations also works in the constant-curvature case without the modification of the gamma matrices, provided one adds to the field equation a suitable combination of terms A_3 and A_4. As a consequence, the quantity $U^{-1} \boxtimes U$ in (57) reduces to the operator $\partial\bar{\partial}$. I believe, however, that the modification of the gamma matrices is a more elegant way to achieve exact solubility.

3.3. Scalar Product

The conserved current associated with the scalar product (S_1, S_2) is given by

$$\begin{aligned} J^k &= \int \bar{S}_1(A^k + B^k) S_2 \, D\theta \, D\bar{\theta} \\ A &= (\bar{\theta}^- \Gamma^k)_m \, \partial \bar{\theta}_m^- + (\Gamma^k \theta^+)_m \, \partial \theta_m^+ \\ B &= 2\hat{\square} \bar{\theta}\theta^+ \\ S_i &= \exp(i\bar{\theta}^+\theta^- + X) T_i; \qquad X = \bar{\theta} \nabla \theta \end{aligned} \tag{64}$$

The term B^k proportional to $\bar{\theta}^-\theta^+$ is a potential source of divergence in the infinite-dimensional theory. For an exponential ansatz this term, however, cancels in a theory based on the modified gamma matrices,

$$(A^k + B^k) \exp(i\bar{\theta}^+\theta^- + X) T = \exp(i\bar{\theta}^+\theta^- + X) A^k T \tag{65}$$

The underlying reason for the vanishing of this term is the commutator relation $B^k = [X, A^k]$ already encountered and the vanishing of the commutator $[X, B^k]$ in the theory defined by modified gamma matrices.

Thus the current can be cast into the form

$$\begin{aligned} J^k &= \int T_1 \exp(iS) A^k T_2 \, D\theta \, D\bar{\theta} \\ S &= (\bar{\theta}\theta - i\bar{\theta}\underset{\sim}{\nabla}\theta + i\bar{\theta}\underset{\sim}{\nabla}\theta) \end{aligned} \tag{66}$$

and has the same general structure as a transition amplitude between states T_1 and T_2 in a field theory described by the free field theory action S.

The scalar product can be evaluated diagrammatically by noticing that:

1. The term $\bar{\theta}\theta$ corresponds to the free field action and gives rise to a propagator that is a unit matrix connecting lines of different chirality. The functional determinant det K coming from the functional integral as a contribution of vacuum bubbles can be taken out of the integral when K is proportional to the unit matrix.

2. The term $i\bar{\theta}\nabla\theta$ is analogous to an interaction term giving rise to a two-particle vertex, where the differential operator $i\nabla$ appears as a vertex factor and connects lines of the same chirality. Note that the differential operator acts on the ingoing or outgoing particles only.

3. The theta parameters appearing in the fields T_1 and $A^k T_2$ correspond to incoming and outgoing particles.

Some general properties of the scalar product should be noticed.

1. The innocent-looking cancelation of the term B^k is of uttermost significance, since it contributes to the scalar product a term proportional to the trace of the matrix K (Feynmann diagram connecting the theta parameters appearing in the factor B^k).

This factor is infinite as a dimension of the spinor basis for the configuration space. Thus, one should give up the simple solution ansatz and replace the matrix K with a more complicated matrix having a finite trace. Then, however, the determinantal factor would become dependent on the surface X and one would face all the divergence difficulties of the conventional regularized quantum field theory when trying to find a definition for the trace and determinant of the matrix K.

2. We found earlier that the addition of a suitable combination of terms A_3 and A_4 to the super-d'Alembertian would reduce the field equations to the purely algebraic condition (56) for constant-curvature spaces. This kind of theory also would be free of divergences. The mechanism canceling potential divergence terms is essentially the same as that which guarantees that the field equations are satisfied. The form of the scalar product is identical to that obtained in the theory based on modified gamma matrices.

3. The appearance of the term $\exp(i\bar{\theta}^+\theta^-)$ in the general solution is necessary in order to obtain a nonvanishing norm for the general solution. On the other hand, the exponential factor can be present in the solution only if the super-d'Alembertian contains only the other half of the theta parameters.

4. The fact that only the second half of theta parameters appears in super-d'Alembertian implies matter-antimatter asymmetry, which manifests itself as a vanishing of the norm for all states containing only antiparticles (the operator A^k annihilates the generating state). The norm is in general nonvanishing for a state containing an arbitrary number of antifermions and at least one fermion.

5. The modification of the super covariant derivatives is necessary in order to obtain an acceptable scalar product. In the theory based on the conventional super covariant derivatives the "action" defining the scalar product would reduce to the term $\bar{\theta}\theta$ as far as M^4 degrees of freedom are concerned.

6. In order to obtain a conserved scalar product, some kind of mass shell condition is necessary. I have already discussed the general form of this condition.

4. SYMMETRIES OF THE THEORY

4.1. Generalized Chiral Invariance

The absence of the super covariant derivatives acting on θ^+ and $\bar{\theta}^-$ implies generalized chiral invariance: the separate conservation of fermion numbers associated with different chiralities of H-spinors (and configuration space spinors also). The interpretation of the conserved fermion numbers as baryon and lepton numbers is natural.

The requirement of nonvanishing norm, however, implies a spontaneous breaking of the generalized chiral invariance. The presence of the exponential term $\exp(i\bar{\theta}^+\theta^-)$ implies that only the quantity $B-L$ is well defined for states having nontrivial norm. Since the separate conservation of B and L is not broken at the level of the field equations, one has good hopes that the nonconservation effects are sufficiently small to guarantee a sufficiently long lifetime for the proton.

One could even hope that the breaking of chiral invariance for individual solutions of the field equations is totally contained in the exponential term: the solution is a product of the exponential term and of a term with well-defined B and L; thus, one could associate well-defined B and L with each state. The study of the pointlike limit of the theory shows that this seems to be possible.

The asymmetry between matter and antimatter at the level of the field equations is the second consequence of the exclusion of the super covariant derivatives acting on θ^+ and $\bar{\theta}^-$. This asymmetry, however, is not reflected in the form of the general solution.

Only the properties of the norm reflect this asymmetry. The norm of a state containing only antiparticles is vanishing. The absence of states containing only antifermions from the spectrum could provide an explanation for the observed matter–antimatter asymmetry. For a given 3-manifold the number of states of net fermion number $N>0$ is larger than the number of states with the net fermion number $-N$. This asymmetry is necessarily present whether the system is in thermal equilibrium or not.

Thus, the observed asymmetry is a remnant of the asymmetry already present in the early phases of cosmic evolution, when antiparticles and particles were in thermal equilibrium.

4.2. Color Symmetry

I shall discuss the color symmetries in considerable detail since the realization of color symmetry has been a source of considerable confusion.

One of the basic ideas of the TGD approach has been the color gravitational analogy, which becomes really manifest in the infinite-dimensional theory, where the local $M^+ \times SU(3)$ becomes the isometry group of the configuration space.

One also can understand the color gravitational analogy as in the Kaluza-Klein approach. Gluon fields on X correspond to the projections of the isometry currents of CP to the 3-surface. The connection defined by these currents makes ordinary irreducible color partial waves covariantly constant. In the pointlike limit one can thus eliminate CP_2 derivatives totally from the covariant derivatives appearing in the field equations by replacing them with the action of the color connection:

$$\partial_k f^m = ij_k^A t_{An}^m f^n \tag{66}$$

Here t_{An}^m is the representation matrix of t_A in the representation defined by partial waves f^m. Of course, this formula holds only for ordinary representations of the color group.

A dangerous feature for the TGD approach is the fact that CP_2 allows only triality zero partial waves. How is it possible to get triality 1 partial waves for quarks?

The attempts to solve this problem have led to the concept of the anomalous hypercharge. CP_2 partial waves are not necessarily ordinary representations of the color group, but can have anomalous color hypercharge, so that triplet partial waves also become possible.

The spinors of CP_2 indeed have intrinsic anomalous hypercharge proportional to their electromagnetic charge ($Y_A = 2Q_{\text{em}}$) and this suggests that the relationship between electromagnetic charge and anomalous hypercharge is completely general.

Thus, it is tempting to identify quarks as particles having integer-valued intrinsic electromagnetic charge but moving in pseudo triplet partial waves. The anomalous hypercharge associated with these waves gives an anomalous electromagnetic charge to the leptonic particle and it behaves like a fractionally charged quark.

The fractionation of electromagnetic charge can be understood by the inspection of formula (66). Since triality $|t| = 1$ pseudo representations of the color group differ from ordinary ones by a phase factor not representable as a function of CP_2 coordinates, one cannot transform the ordinary CP_2 derivative to a mere connection term, but an additional term proportional to the Kähler potential appears on the right-hand side of equation (66).

The effect is a change in the structure of the spinor connection. As shown in previous work (Pitkänen, 1983, 1985), the spinor connection of CP_2 (or rather its projection to the 3-surface) can be identified with the electroweak gauge field, and the additional term proportional to the Kähler potential changes the coupling structure so that fractional electromagnetic charge results.

I have suggested several different scenarios to describe quarks and leptons in the TGD framework. The first scenario (Pitkänen, 1983) was based on the assumption that different chiralities of H-spinors correspond to quarks and leptons and the intrinsic hypercharge of quarks is fractional. This scenario is in contradiction with the idea that triplet partial waves carry anomalous Q_{em}. Of course, the separate conservation of B and L is easy to explain.

The second scenario (Pitkänen, 1985) is based on the idea of the anomalous hypercharge and on the assumption that quarks are simply leptons moving in $|t|=1$ partial waves.

The final scenario (I hope so!) is in some sense a compromise between the previous two scenarios: quarks and leptons correspond to different chiralities of H-spinors and have leptonic intrinsic charges, and quarks move in $|t|=1$ partial waves. The separate conservation of B and L is spontaneously broken.

The above physical picture can be put on a sound mathematical basis. The partial wave analysis in CP_2 (Pope, 1980, 1982) can be solved exactly for all important differential operators defined in CP_2 (Dirac operator, wave equations associated with various spins).

The main results are;

1. The problem can be reduced to the solution of a d'Alembert equation for a scalar field coupled to the Kähler potential via an appropriate charge s, which depends on an individual solution.

2. The d'Alembert equation for the charged scalar particle is exactly soluble and the solutions correspond to irreducible representations of the color group. When the scalar field carries Kähler charge s these representations are of the type $(p, p+s)$ $(s>0)$ or $(P-s, P)$ $(s<0)$. Thus, the concept of a $t=1$ representation with an anomalous hypercharge is indeed completely well founded mathematically.

3. The solutions of the Dirac equation carry same total anomalous hypercharge. The anomalous hypercharges of color partial waves thus compensate the different intrinsic hypercharges associated with spinor components with different electromagnetic charge.

This result is in contradiction with the idea that anomalous hypercharge is proportional to electromagnetic charge and indeed leads to an unphysical consequence. Only the states having the quantum numbers of a right-handed neutrino or a right-handed d-quark move in the expected color representations. For example, the charged leptons move in decouplet color partial waves (Pope, 1980, 1982).

The last result is very important since it favors very strongly the scenario where the CP_2 Dirac operator is replaced with a modified Dirac operator having an identically vanishing square. As a consequence, we can choose the spinor basis so that all spinors are solutions of the same scalar d'Alembertian with an appropriate anomalous hypercharge, in accordance with the idea that the total anomalous hypercharge is proportional to electromagnetic charge. Nothing forbids the spinors of different chiralities to move in $t=0$ and $|t|=1$ partial waves, respectively.

The fractionation of the electromagnetic charge and the appearance of $|t|=1$ pseudorepresentations can be understood in a mathematically rigorous manner in terms of central extension for the Lie algebra of the color group. The replacement

$$j\cdot\nabla\rightarrow j^k D_k \tag{67}$$

where D_k is the covariant derivative defined by the Kähler potential, in the differential operator representation of the Lie algebra of the color group leads to a new Lie algebra, which is the central extension of the original algebra. Thus, the pseudorepresentations correspond to ordinary representations of the centrally extended $SU(3)$.

It is important to emphasize that the choice of the particular central extension is not dictated by the particular central extension appearing in the field equations, and thus it is possible to obtain both $t=0$ and $|t|=1$ states, although the spinor connection of the configuration space corresponds only to the $t=0$ representations.

In fact, the couplings of the quarks to electromagnetic charge can be made fractional by the direct construction of bosonic charge matrices, as the study of the pointlike limit of the theory shows. Therefore, we believe that problems related to color and electromagnetic charge are finally solved.

4.3. Super Gauge Invariance and Supersymmetry

Any superfield generated from a scalar function is a vacuum solution with zero norm and one can always add to the superfield this kind of solution. I called this symmetry Abelian super gauge invariance in (Pitkänen, 1986). Clearly, the function of this symmetry is to eliminate the states generated by pure scalars from the theory.

A more interesting symmetry is $N=1$ global supersymmetry of the pointlike limit of the theory. In order to understand the emergence of the $N=1$ global supersymmetry, consider three properties of the right-handed neutrino spinor;

1. Covariant constancy of the right-handed neutrino and its conjugate.

2. The right-handed neutrino is annihilated by the modified gamma matrices Γ_+^k. The Dirac conjugate of the right-handed neutrino is annihilated by the matrices Γ_-^k.

3. Quark lepton orthogonality allows both H-chiralities for the right-handed neutrino, as will be found in the sequel. This implies that supersymmetry acts on the quark and lepton sector of the theory.

The $N=1$ supersymmetry is present also in the complete theory provided the "vacuum" spinor is covariantly constant. The possible covariant constancy of this spinor depends on the nature of the central extension defining the spinor connection of the configuration space. It might be that for a suitable choice of the integers appearing in the Kähler potential the covariant constancy is possible, although I have no proof for this. The requirement of $N=1$ supersymmetry would thus fix the spinor connection of the configuration space completely.

The covariant constancy implies that the quantities

$$A=\bar{\theta}^+U; \qquad B=V\theta^- \tag{68}$$

formed as contractions of the spinors U and V having the quantum numbers of right-handed neutrino with the "wrong" thetas behave as constant with respect to the super covariant derivatives, and one can obtain the super partner of a given state S by multiplying it with the quantities A and B. The condition $\partial\bar{\partial}T=0$ holds identically for the state obtained in the symmetry.

Note that the generalized chiral invariance and covariant constancy of the right-handed neutrino are necessary for this symmetry. No assumptions concerning the form of the Dirac operator are needed.

The operation can be interpreted physically as the addition of a right-handed antineutrino of either chirality to the state. We shall find that when the particle in question has unphysical helicity this operation indeed leads to a super partner of the original particle. Otherwise, the particle number of the state increases by one unit.

When modified gamma matrices appear in the super-d'Alembertian and annihilate the CP_2 spinor with quantum numbers of the right-handed neutrino (in fact, the right-handed neutrino can be defined in this manner), also the multiplication with the quantity

$$\bar{A}=\bar{U}\theta^+ \tag{69}$$

and its reverse as supersymmetries provided the new state satisfies the condition $\partial\bar{\partial}T=0$.

In addition, the right-handed neutrino must be solution of the M^4 Dirac operator $\not{P}$ appearing in the field equation and thus corresponds to a fermion of unphysical helicity. Thus, this symmetry generates only true elementary particle states. Observe that this supersymmetry is not present in a theory defined by ordinary gamma matrices.

This supersymmetry acts only in the leptonic sector. This is due to the fact that the conjugate spinor of a right-handed neutrino is not annihilated by the modified gamma matrices. This asymmetry might be the ultimate reason for the different color properties of the two associated with two H-chiralities.

Associated with $N=1$ supersymmetry is so-called R-invariance (Stelle, 1983). The number of right-handed neutrinos of either chirality is conserved. One can associate with each particle state a definite R-chirality and ordinary particles are identified as $R=0$ particles.

This concept has proved very useful in the construction of the solutions of the $M^4\times CP_2$ super-d'Alembertian and in finding the physical interpretation for the individual elements of the basis. In particular, the ordinary electroweak gauge bosons correspond to $R=0$ states and spin-one particles with nonstandard couplings correspond to $R=1$ states.

4.4. Local $M^4\times SU(3)$ Symmetry[3]

The enormous isometry group of Map(X, H) is expected to simplify decisively the construction of the physical states and the calculation of the S-matrix elements and vertices. I shall now show that the state construction leads to a formalism resembling the quantization of string models (Goddard-Olive, 1983).

The constant-curvature property makes it possible to simplify calculations decisively. The general form of the Kac-Moody algebra is universal in the basis where extension is diagonal; the diagonal elements of central extension are proportional to integers. Thus, the construction of the representations can be carried out in general form without worrying about the details of the scalar function basis.

The more detailed treatment necessitates the evaluation of the degeneracy of scalar functions associated with a given value of the diagonal element of the central extension. This degeneracy certainly depends on surface X, since 3-surfaces allow nonequivalent magnetic structures.

[3]It is probable that the local M^4 symmetry in fact extends to local Poincaré symmetry. This is in accordance with the idea that local Poincaré group is the gauge group of gravitational interactions.

Mimicking the state construction in string models (Goddard-Olive, 1983; Kac, 1983), we assume:

1. There exists a vacuum state $|0>$, which is (a) annihilated by the diffeogenerators corresponding to positive eigenvalues of the differential operator D defining the interior part of the central extension,

$$d_m|0\rangle = 0; \qquad m > 0 \tag{70}$$

(b) annihilated by the Kac–Moody generators with positive eigenvalues,

$$J^A_m|0\rangle = 0; \qquad m > 0 \tag{71}$$

(c) an eigenstate of the diffeogenerator D,

$$D|0\rangle = h|0\rangle \tag{72}$$

(d) corresponds to an irreducible representation of the isometry group of H.

2. Physical states are generated by polynomials of the Kac–Moody generators J^A_m, $m > 0$, of the type

$$|\text{Phys}\rangle = \prod_{m_i} J^{A_i}_{m_i}|0\rangle; \qquad m_i > 0 \tag{73}$$

3. Physical states are (a) annihilated by the diffeogenerators with negative eigenvalues and (b) eigenstates of the generator D; the eigenvalue is the same for all states in the representation.

This assumption does not necessarily imply any diffeoanomaly, since only a trivial central extension of the Diff Lie algebra leads to this condition, as will be found soon.

Vacuum and physical states are not annihilated by the positive eigenvalue diffeogenerators. The states obtained in this manner have, however, vanishing norm provided diffeo- and Kac–Moody generators with opposite eigenvalue are Hermitian conjugates of each other:

$$(J^A_m)^\dagger = J^A_{-m} \tag{74}$$

This condition ensures that the action of the diffeomorphism generators to the physical states creates only zero-norm states. The possible existence of a nontrivial central extension of Diff (diffeo anomaly) spoils the zero-norm property.

4. The irreducibility of the representation requires that physical states are eigenstates of some Casimir-type operator. Field equations do not, however, fix this operator uniquely. Thus, although the metric of G/F is determined by a unique central extension, the solution of the field equations

does not exclude the possibility that the Casimir operator belongs to a different central extension than the one defining the spinor connection. In the finite-dimensional case we shall find that leptonic and quarklike states correspond to different central extensions in CP_2.

The simplest guess is that states are "massless" in the sense that the Casimir operator naturally defined by the super-d'Alambertian annihilates them. In the general case this operator gives the Casimir operator of G/F.

When the Dirac operator appearing in the field equations is defined by the modified gamma matrices, this operator reduces to the ordinary M^4 d'Alembertian! The physical states would be massless. This result is not as crazy as it looks, since the mass scale of ordinary elementary particles is extremely small compared with the Planck mass scale associated with CP_2. In this sense the masslessness condition is a good starting approximation.

Irreducibility does not, however, imply the vanishing of the mass squared operator. This suggests a possibility of understanding the mass scale problem: require only that the mass squared operator is proportional to the eigenvalue of the centrally extended Casimir operator of G/F minus the Casimir operator of $M^4 \times CP_2$. The value 10^{38} for the proportionality constant makes the states approximately massless. The freedom to choose the value of the string tension in the string model is the mathematical equivalent of this freedom.

What fixes the value of the elementary particle mass scale? Perhaps the convergence requirement of the exponential expansion of the solution of the super-d'Alembertian leads to mass quantization. Also, the requirement that the continuation of the single-particle superfield in the whole configuration space is single-valued might lead to mass quantization.

In the quantization of the string model one encounters the so-called conformal anomaly; the Virasoro algebra (generating holomorphic maps of the string to itself) suffers a nontrivial central extension. Could something like this also happen now?

One can extend the central extension to an extension of the Lie algebra of the diffeomorphism group in an obvious manner,

$$[X, Y] = [X, Y] + k \int [j, X] Y \, dx \, Id + \text{boundary terms} \tag{75}$$

Here the scalar product is the usual scalar product for the vector fields defined in X. This extension is, however, trivial and by a suitable redefinition of the diffeogenerators (add a suitable multiple of the unit matrix to the original diffeogenerators) of the generator basis the commutation relations reduce to the original ones. Whether one can find a nontrivial extension is an open question. It might well be possible to construct the general form of possible extensions by applying considerations similar to those used in the string model (Goddard-Olive, 1983).

5. POINTLIKE LIMIT OF THE THEORY

5.1. Definition of the Pointlike Limit

The most obvious guess for the pointlike limit of the theory is the theory defined by the super-d'Alembertian of H. A closer inspection shows that the limiting procedure need not be so simple.

The point is that all color partial waves are possible for the H super-d'Alembertian, suggesting that ordinary elementary particles move in all possible CP_2 partial waves. This result is in disagreement with the conventional ideas about color.

On the other hand, the general properties of the Kac–Moody representations suggest that only a finite number of CP_2 partial waves is possible. The representations of the Kac–Moody group are known to correspond to the so-called highest weight representations of the Kac–Moody algebra (Goodman and Wallach, 1985) and these satisfy an important constraint, which I now describe.

Each representation can be generated from a vacuum vector (described in the context of symmetry considerations) transforming according to some irreducible representation of $SU(3)$ with highest weight $\bar{\lambda}$. The vacuum representation corresponds clearly to the representation associated with the multiplet of CP_2 partial waves associated with the degrees of freedom that appear in the super-d'Alembertian at the pointlike limit.

This representation is not arbitrary, however. The integer k defining the central extension satisfies the following lower bound:

$$k \geq |\bar{\lambda} \cdot \bar{\alpha}| \tag{76}$$

where $\bar{\alpha}$ is some root vector associated with the Lie algebra of $SU(3)$ (Goodman and Wallach, 1985).

Thus, only a finite number of CP_2 partial waves are possible for a generic surface and the number of possible partial waves can be easily evaluated. Somehow, color is forced into the vibrational degrees of freedom. It should be emphasized, however, that the generating solutions of the super-d'Alembertian might correspond to a central extension with arbitrarily large integer k.

Thus, we can conclude that the various CP_2 partial waves appearing in the pointlike theory correspond to different irreducible representations of the Kac–Moody algebra. Furthermore, if only a single central extension is allowed, then only a few partial waves correspond to real states and the super-d'Alembertian of H does not give a reliable description of the pointlike limit.

A second argument against this simple-minded description of the pointlike limit is that configuration space spinors are in correspondence with the scalar function basis of X. For magnetically trivial 3-surfaces one can localize configuration space spinors to boundary components; matter resides on boundaries. In the pointlike limit, however, all these spinors disappear from the theory. Thus, one expects that higher fermion families disappear from the fermion spectrum in the naive pointlike limit.

Consider now the problem of finding a definition of the elementary particle solution. The interpretation of all solutions as elementary particle states is certainly ruled out, since it leads to the prediction of elementary particles of very high lepton and baryon numbers (as high as four). Also, spins larger than two are allowed. A possible solution to this problem is afforded by the multiplicative superposition property of the solutions of the field equations. If T_1 and T_2 generate solutions to the field equations, then also their product generates a solution of the field equations provided the condition $\partial\bar{\partial}T_1T_2=0$ is satisfied. In fact, the solution generated by the product is the product of the solutions.

Thus it is natural to define elementary particle solution as a solution that is not expressible as a superposition of products of more elementary solutions. If the condition $\partial\bar{\partial}T_1T_2=0$ were satisfied for all elementary particle pairs, physical states would form an algebra generated by elementary particle states. In fact, quarks and leptons with physical helicity satisfy the condition and indeed generate an algebra of many-fermion states. This is not true generally, however.

The following principles are of considerable help in the construction of elementary particle states.

1. Elementary particle states have well-defined baryon and lepton numbers in the sense that the fields T generating them must have well-defined B and L. Furthermore, states with different B and L must be orthogonal to each other.

2. Elementary particle states are expressible in terms of various observables related to physical particles (polarization vectors and tensors, charge matrices, H-parities, etc.).

3. $N=1$ global supersymmetry with associated R-invariance can be used to divide the elementary particles into ordinary ones ($R=0$) and their super partners ($R=+1$). For example, only spin-one bosons having charge matrix typical for electroweak gauge bosons have $R=0$, and charged spin-1 bosons coupling to right-handed fermions are $R=1$ states.

4. Elementary particle states can be regarded as superpositions of four types of fermions:

(a) Physical fermions and antifermions annihilated by the operator $\hat{\not{P}}$,

$$\hat{\not{p}}u = 0; \qquad (\hat{p}_0, \hat{\bar{p}}) = (p_0, -\bar{p}) \tag{77a}$$

These particles generate many-fermion states. The quark lepton orthogonality is identically satisfied for these states; $qL = 0$.

(b) Unphysical fermions and antifermions annihilated by the operator $\not{p}$,

$$\not{p}u = 0 \tag{77b}$$

Together with physical fermions, these states generate elementary particles. For example, gauge bosons can be regarded as states of type $\bar{L}_u L_f + \bar{q}_u q_f$.

5. The norm of a physical state must be nonvanishing. A useful observation with regard to the evaluation of the norm of the various candidate states is that in the norm the unphysical antifermion $\bar{q}_u/\bar{L}_u$ must always contract directly with a physical fermion L_f/q_f.

For nonvanishing terms in the scalar product the unphysical fermions L_u/q_u either appear as the Dirac scalar product term $\bar{F}\Gamma^k F$ (only one of them) or are directly contracted with a physical antifermion $\bar{q}_f/\bar{L}_f$.

These requirements can be satisfied if:

1. Corresponding to each $\bar{L}_u/\bar{q}_u$, the state contains q_f/L_f with the same weak isospin. Example: the leptoquark states of type $\underline{\bar{L}_u q_f}$ and $\underline{L_f \bar{q}_u}$, where the bracket implies that the weak isospins are the same.

2. The state is a suitable superposition of many-particle states. Example: the bosons are of type $\bar{L}_u L_f + \bar{q}_u q_f$.

Although the representation of states in terms of physical and unphysical fermions is not very illustrative, I believe that using this representation one could show that elementary particles satisfy certain generally accepted constraints:

1. $B - L$ is equal to 0, +1, −1; L and B are equal to 0, +1, −1.
2. The largest value of spin and weak isospin is 2.

In the sequel I shall show that the spectrum contains particles identifiable as known fermions and bosons. Only one fermion family is found, in accordance with the idea that the family replication phenomenon has a topological origin. Of course, the configuration space spinors corresponding to more than one creation operator might provide an alternative explanation for family replication.

5.2. Fermions

One-fermion solutions of different chiralities will be identified as quarks and leptons. Both leptonic and quarklike spinors are representable as products of the M^4 spinor u and the CP_2 spinor v, and the M^4 spinor is annihilated by the matrix $\hat{p\!\!/}$ defined in (5.2).

The requirement that all fermions are annihilated by a negative-energy Dirac operator is essential for obtaining a nonvanishing norm for antifermions, to guarantee quark lepton orthogonality, and to give well-defined M^4 helicity for the fermion.

Leptonic particles correspond to products of constant CP_2 spinors with $t=0$ partial waves in the (p, p) representations of $SU(3)$.

For quarks the CP_2 spinor is assumed to have the following representation:

$$U = d_+ U_0, \qquad D = d_- D_0 \tag{78}$$

The quarklike spinors U_0 and D_0 are products of leptonic spinors with $|t|=1$ pseudo partial waves in CP_2.

With the above choice, quarks are indeed mutually orthogonal. Only the scalar products of left/right-handed quarks need to be considered and u- and d-quarks are orthogonal because the square of the modified Dirac operator d_+/d_- vanishes.

In order to obtain left-handed u/d-quarks, one must choose just d_- and d_+ in the above equations, since the modified gamma matrices appearing in d_+ annihilate identically right-handed u spinors of CP_2.

Observe that the absence of the d operators in the definition of leptonic states is necessary: otherwise, one would not obtain the left-handed neutrino, due to the fact that both d_+ and d_- annihilate the CP_2 spinor with the quantum numbers of the right-handed neutrino.

The right-handed neutrino is exceptional in the sense that quark-lepton orthogonality allows both chiralities for it. This is due to the covariant constancy of the right-handed neutrino and the fact the quark superfields are proportional to the operators d_+ and d_-. As a consequence, the scalar product between quarks and right-handed neutrinos of both chiralities vanishes. This specific feature of the right-handed neutrino is essential in guaranteeing that the supersymmetry of the theory is consistent with the different assignments of color representations to different chiralities.

An important feature of the construction is that quarklike spinors must move in triality-one partial waves. Otherwise, the lowest left-handed u-quark would disappear from the spectrum, since the action of covariant derivatives annihilates the covariantly constant right-handed d-spinor.

5.3. Bosonic spectrum

5.3.1. Spin-One Bosons with Zero Fermion Numbers

We define spin-one bosons as particles characterized by a polarization vector either in Minkowski space or in CP_2. The requirement of nonvanishing norm implies the usual conditions for the Minkowski space polarization vector,

$$P \cdot e = 0 \qquad e \cdot e < 0 \tag{79}$$

Thus, only two polarizations are possible for massless states.

When the polarization is in the direction of CP_2 it is natural to assume that the polarization vector is proportional to one of the isometry currents of CP_2 in order to obtain a quantity with well-defined color transformation properties.

Spin-one particles are either vectors or axial vectors in the generalized sense:

$$V = \bar{\theta} \not{e} Q \theta \tag{80a}$$

$$A = \bar{\theta} \not{e} Q \Gamma_9 \theta \tag{80b}$$

These states are mutually orthogonal, for obvious reasons. The axial vectors are not found in the spectrum of observed particles. A nice explanation would be that the spontaneous breaking of the generalized chiral invariance makes these particles massive, as happens in the breaking of the ordinary chiral invariance.

The couplings of spin-one particles to physical and unphysical helicities are of the same or opposite sign. In terms of the helicity projectors

$$P_+ = 1, \qquad P_- = [\not{p}, \hat{\not{p}}]/2p_0 \tag{81}$$

the charge matrices can be written in the form

$$Q = Q_0 \otimes P_\pm \tag{82}$$

where the matrices Q_0 are superpositions of CP_2 sigma matrices.

The four types of bosonic states are orthogonal to each other and have positive norm. The requirement that the state is orthogonal to an arbitrary quark lepton state excludes the bosons of type V_+ and A_-. Thus, the physical gauge bosons are of type V_- and A_+.

Consider next the charge matrices associated with spin-one bosons. There are altogether eight charge matrices associated to left- and right-handed spinors of CP_2. Thus, we must understand why the observed boson spectrum does not contain spin-one charged particles with right-handed couplings.

The $N = 1$ supersymmetry provides a solution to this problem. The states corresponding to nonallowed charge matrices contain one index coupling to the right-handed neutrino and must be regarded as states obtained via supersymmetry from their fermionic counterpart and having

R- parity different from zero. Note that this argument does not exclude fractional charges for quarks, since in this case the electromagnetic charge matrix contains a part proportional to the tensor product of the right-handed neutrino and its antiparticle: R-parity is zero.

Thus, $N=1$ supersymmetry leaves only the charge matrices associated with the standard model. The requirement that the photon and Z^0 are orthogonal implies the orthogonality of the charge matrices and the value $\sin^2\theta_w = 9/26$ for the Weinberg angle (Pitkänen, 1983).

5.3.2. Spin-two particles

Spin-two particles can be constructed as two-particle states constructed from spin-one bosons. At least one of the spin-one bosons must be unphysical; otherwise, the state would represent a two-particle state of two physical bosons.

The second boson must be physical. This assumption guarantees orthogonality to the states containing two quarks and two leptons. Thus, only the following types of spin-two particles are physical:

$$V_+V_-;\quad V_+A_+;\quad A_+A_-;\quad A_-V_- \tag{83}$$

The graviton must correspond to a particle of type VV or perhaps of a superposition of VV and AA, for obvious reasons. The breaking of the generalized chiral invariance is expected to make the particles of type VA and AV (and perhaps AA) heavy.

The fact that V_- couples with different signs to different spinor helicities is essential to guarantee the orthogonality of the state with respect to the two-photon state, when a fractional charge matrix is allowed for photons. The scalar product involves traces over the charge matrix and these vanish only due to the helicity properties of V_-. I have found no argument to exclude charged gravitons. Of course, the electroweak symmetry-breaking is expected to make them massive.

The color gravitational analogy suggests strongly that the gluon and the graviton have similar tensor structure. Gluons should simply correspond to particles of type VV where the polarization vectors are given by $\not{e}$ and $\not{j}_A$, with j_A denoting a generator of CP_2 isometry.

5.3.3. Leptoquarks

One can construct also bosons with nonvanishing baryon and lepton numbers, which are expected to be massive for obvious reasons. These states are assumed to have definite baryon and lepton numbers and to have the following general structure:

$$A = \bar{\theta} X \theta^+$$
$$A^+ = \bar{\theta}^+ X^+ \theta^- \tag{84}$$
$$X = Pv \times Q_0 \times P_\pm$$

Couplings to different helicities are of opposite sign in order to guarantee orthogonality to lepton–lepton and quark–antiquark states.

The polarization tensor *Pv can* have at least some of the following forms:

$$[\not{e}_1, \not{e}_2]; \quad J_{kl}\Sigma^{kl}, \quad [\not{j}_A, \not{j}_B]; \quad [\not{e}, \not{j}_A]$$

The charge matrices could be the ones appearing in the representations of ordinary spin-1 bosons.

We expect that the unphysical leptoquarks and spin-1 bosons together give rise to a multitude of particles, constructed typically from one unphysical and one physical particle.

6. CONSTRUCTION OF S-MATRIX

6.1. General Considerations

The multiplicative superposition suggests the following formal procedure for the construction of a unitary S-matrix.

1. Construction of bare one-particle states. The field equations are solved in the sets Map(t, H), where t denotes the connected manifold topology. We define the bare one-particle states as state functionals restricted to Map(t, H) and having the property that they are stationary solutions of the field equations in Map(t, H).

In this context the meaning of the bare one-particle state is rather general; only the simplest 3-topologies are expected to correspond to elementary particles and the topologies obtained by forming connected sums of the simple topologies (say, by gluing particlelike 3-manifolds to a subset of a spacelike hyperplane of M^4) are expected to correspond to (gravitationally or otherwise) bound states of elementary particles.

We assume that the bare one-particle states can be orthonormalized with respect to the conserved scalar product implied by the phase symmetry of the action and that this scalar product is positive definite, at least, when restricted to the set of "physical states." Furthermore, single-particle state functionals are assumed to form a complete set with respect to this scalar product.

2. Construction of bare many-particle states. By multiplicative superposition the products of one-particle state functionals solving the field equations in Map(t_n, H) are solutions of the field equations in Map(t_n, H)

(set of n-component 3-manifolds with fixed topologies). It is natural to define the bare n-particle states as state functionals that vanish outside $\text{Map}(t_n, H)$ and are superpositions of the products of one-particle state functionals in $\text{Map}(t_n, H)$. The scalar product for single-particle states defines a natural scalar product for bare many-particle states.

3. Construction of stationary states. Bare n-particle states are solutions of the field equations both inside and outside (trivially so) $\text{Map}(t_n, H)$, but not in the boundaries of $\text{Map}(t_n, H)$. Thus, these states are not stationary and by the uncertainty principle are expected to disperse to the other parts of the configuration space. This dispersion is observed as various particle reactions, which may change particle number and also the topological quantum numbers associated with a single particld. By multiplicative superposition the stationary states must correspond to linear superpositions of bare n-particle states.

One can construct from an n-particle bare state a stationary state by continuing this state to the other topologically different sectors of the configuration space. The continuation of the bare n-particle state to a stationary state must be one-valued. This requirement probably poses constraints on the spectrum of allowed bare states and might well lead to quantization conditions.

4. Definition of S-matrix. Bare n-particle states resemble the incoming states of the ordinary field theories; they have sharp particle number and they are not global solutions of the field equations. The stationary states in turn resemble the outgoing states, since they are stationary solutions of the field equations and have no sharp particle number. Thus, it is natural to define the S-matrix as a matrix transforming the bare and stationary states into each other.

In the following subsections we shall:

1. Consider the problem of comparing superfields associated with topologies t_1 and t_2 in the singular limit $t_1 \to t_{12} \leftarrow t_2$.

2. Formulate the continuity conditions making it possible to continue a given bare state functional to a stationary state functional.

3. Derive a formal solution to the continuity conditions in terms of certain overlap integrals over $\text{Map}(t_{12}, H)$. As a result, one obtains explicit expressions for couplings (say, the electromagnetic coupling!) as overlap integrals over $\text{Map}(t_{12}, H)$.

4. Derive from the one-valuedness requirement of the stationary state functionals a set of conditions analogous to the conditions defining the duality concept familiar from the string models (Jacob, 1974; Chew and Rosenzweig, 1978).

5. Derive a general expression for the S-matrix and discuss the problem of the practical evaluation of S-matrix elements.

6.2. Description of the Limiting Procedure

The continuity conditions state that the limiting values of the superfields associated with two manifold topologies t_1 and t_2 in the limit are equal in the limit of the singular manifold topology: $t_1 \to t_{12} \leftarrow t_2$.

To relate the superfields defined in different parts of the configuration space, we must be able to define the limiting procedure

$$\mathrm{Map}(t_i, H) \to \mathrm{Map}(t_{12}, H), \qquad i = 1, 2$$

in a unique manner. This limiting procedure defines a representation for the geometry of the space $\mathrm{Map}(t_{12}, H)$ and thus the calculation of various overlap integrals encountered in the construction of the S-matrix can be performed using the corresponding G-invariant integration measure.

Of course, the limits $t_i \to t_{12}$ give rise to two different representations of the geometry of $\mathrm{Map}(t_{12}, \mathrm{H})$ but these representations must correspond to two different choices of coordinates for t_{12} (different choices of scalar function basis).

The fundamental topology changes correspond either to a change in the internal topology of a connected manifold or to a decay of a connected manifold. Thus, we shall restrict our attention to these cases from now on.

A convenient way to describe the limiting procedure is via the following trick. Let d be a singular diffeomorphism, which maps X with initial topology to a singular manifold with intermediate topology t_{12}. For instance, d could map:

1. A sphere to a manifold that is intermediate between a sphere and two disjoint spheres; in this case d maps some 2-surface of X to a single point.

2. A surface $S^1 \times S^2$ to a manifold with topology intermediate between a torus and S^3-topology; in this case d could map the 2-surface $p \times S^2$ to a point for some point P of S^1.

3. A manifold with one boundary component to a singular manifold intermediate between the original manifold and a manifold with two boundary components.

If $\{s_m\}$ is the scalar function basis associated with the initial (or final) topology, the scalar function basis associated with the intermediate topology is simply the basis $\{s_m \circ d\}$. The surfaces obtained by exponentiating these Lie-algebra generators indeed describe singular surfaces. The singularity surface associated with the map s is always mapped to a single point in H.

Since the complexified scalar function basis plays a central role in the definition of the metric and related quantities, this correspondence makes it possible to define the limiting procedure for the metric, the vielbein, and connections.

In order to compare superfields associated with t_1 and t_2, we must be able to relate the spinors and theta parameters associated with different manifold topologies and with the intermediate manifold topology to each other.

The Hilbert space of spinors is "universal" and the theta parameters are in one-to-one correspondence with the annihilation and creation operators defining this space. Thus, the integration measure associated with the theta parameters is also universal. All that is needed is to find a natural representation of the initial and final gamma matrices in the universal algebra of annihilation and creation operators.

What is involved in this limiting procedure is best described by a concrete example. Let us consider the decay of a connected 3-manifold into two components X_1 and X_2 via the intermediate singular manifold with one point of X_1 and X_2 identified.

The scalar function basis $\{s_m\}$ associated with X goes to the scalar function basis $\{s_m \circ d\}$ (d is a map contracting the 2-surface of X to a point) associated with the intermediate topology. The restriction

$$s_m \circ d_{|X_i}, \qquad i = 1, 2$$

of the scalar function basis defines a natural scalar function basis for X_i. The direct sum of these bases gives an alternative scalar function basis for X_{12}. Thus, one can evaluate the limits of the geometric quantities.

One must imbed the bare gamma matrices associated with X_1 and X_2 to suitably chosen disjoint subalgebras of the universal algebra and require that the imbeddings together span the whole algebra. For instance, one can associate the complexified gamma matrices associated with X_1/X_2 with the operators a_n and a_n^+ with even/odd index n.

Since the scalar functions associated with the initial topology are sums of their restrictions to X_1 and X_2, the gamma matrices associated with X are sums of the gamma matrices associated with X_1 and X_2. Hence, the bare gamma matrices associated with the initial topology correspond to the operators

$$(a_n + a_{n+1})/\sqrt{2}; \qquad (a_n^+ + a_{n+1}^+)/\sqrt{2}$$

Thus, we have obtained what is needed to compare the superfields associated with the initial and final topologies, and can calculate the typical overlap integrals of three superfields defined in X_1, X_2, and X_{12}.

Notice that the exponential factors $\exp(i\bar{\theta}^+\theta^-)$ associated with the final states must be replaced with the factors $\exp(i\bar{\theta}^+P\theta^-)$, where P projects to the subspace of even/odd thetas. As a consequence, the exponential factors combine to give the conjugate of the corresponding factor associated with the inital state and one can evaluate overlap integrals using the same rules as used in the calculation of ordinary scalar products.

6.3. Formal Solution of the Continuity Conditions

In order to continue a given bare state functional defined in Map(t, H) to a stationary state functional, one can use the continuity requirement of the state functional in the sets Map(t_i, H). In this subsection I shall formulate the continuity conditions and derive a formal solution of the conditions.

Let $V_i(t)$ denote a basis of bare state functionals defined in Map(t_i, H), $i=1,2$. The phase symmetry of the super-d'Alembertian implies the existence of a scalar product, which we assume to be positive definite in the subspace of physical states. Thus, we can assume that bare states form a complete, orthonormalized set.

With these preliminaries we are ready to derive a formal solution to the continuity conditions. The continuity conditions state that for two "neighboring" topologies ti, $i=1,2$, the orthonormalized state functionals belonging to $V(t1)$ $[V(t2)]$ are expressible as a linear combination of the corresponding state functionals belonging to $V(t2)$ $[V(t1)]$,

$$S^m(ti)=\sum_n S^n(tj)G^{nm}(tj, ti), \qquad i\neq j \tag{85}$$

These conditions can be expressed in a more concise form using matrix notation,

$$S(ti)=S(tj)\circ G(tj, ti), \qquad i\neq j \tag{86}$$

The components of the matrices $G(t1, t2)$ and $G(t2, t1)$ are the unknown quantities we wish to solve.

The matrices $G(ti, tj)$ and $G(tj, ti)$ are inverse matrices in the sense that the following equations hold:

$$G(ti, tj)\circ G(tj, ti)=Id(ti) \tag{87}$$

In order to solve the components of the unknown matrices from the continuity conditions (85), we multiply them with a given state functional $S^m(ti)$ of $V(ti)$ and perform the integral over the theta parameters and over Map(t_{ij}, H) (overlap integral over singular manifolds). Define the matrices $H(ti, tj)$ $(i,j=1,2)$ by the following formula:

$$\begin{aligned} H^{mn}(ti, tj) &= (S^m(ti), S^n(tj)) \\ &= \int \bar{S}^m(ti)S^n(tj)DV(t_{ij}) \end{aligned} \tag{88}$$

where the scalar product is defined by the integration measure, which is the product of the integration measure of Map(t_{ij}, H) and the measure associated with the Grassmann algebra.

Using these definitions, one can cast the continuity conditions into the following form:

$$H(tj, ti) = H(ti, tk) \circ G(tk, ti) \tag{89}$$

Using equation (87), it is easy to verify that only two of these equations are independent of each other. For example, the equations corresponding to index pairs $(i, j) = (1, 1)$ and $(2, 2)$ imply the remaining equations.

If the matrix $H(ti, tj)$ is invertible in the sense that there exists a matrix $I(tj, ti)$ with the property

$$I(ti, tj) \circ H(tj, ti) = Id(ti) \tag{90}$$

then the unknown matrix $G(ti, tj)$ can be solved from (6.5) and written in the following form:

$$G(tj, ti) = I(tj, tk) \circ H(tk, ti) \tag{91}$$

Thus, we have expressed the matrices $G(ti, tj)$ in terms of overlap integrals of the bare state functionals over the set of singular manifolds, which are in principle calculable.

The assumption about the invertibility of the matrix $H(ti, ti)$ is clearly a crucial step in the formal solution of the continuity conditions.

6.3. One-Valuedness Requirement

The continuation of the bare n-particle state functional $S^m(t)$ [restricted to Map(t, H)] to a stationary state functional $S_S^m(t)$ (having no sharp particle number) can be performed by applying the formal solution of the continuity conditions. Thus, the stationary state functional can be written as a sum of bare state functionals

$$S_S^m(t) = S^m(t) + \sum_{j,n} S^n(tj) G^{nm}(tj, t) \tag{92}$$

Here the matrices $G(t, tj)$ can be decomposed into products of the matrices $G(t, tj)$ associated with the continuations between "neighboring" 3-manifold topologies (there exist 3-surfaces having topology intermediate between two 3-manifold topologies):

$$G(tj, t) = G(tj, t1) \circ G(t1, t2) \cdots \circ G(tm, t) \tag{93}$$

In general it is kinematically possible (the intermediate states in the continuation are "on-mass-shell states") to perform the continuation $t \to tj$ via several paths and each of these paths must lead to the same final result. The uniqueness of the final result is guaranteed if the product of the matrices $G(tj, t)$ associated with a given path of continuation depends only on the initial and final topologies.

Equivalently, the product of matrices G associated with a closed kinematically allowed path of continuations $t \to tm \to \to \cdots t1 \to t$ is always a unit matrix;

$$G(t, t1) \circ G(t1, t2) \cdots \circ G(tm, t) = Id(t) \tag{94}$$

One might expect that the continuations are strongly restricted by the kinematical constraints, since the intermediate states of the continuation must be solutions of the field equations: thus, all particles in intermediate states must be on-mass-shell particles. The general solution of the super-d'Alembertian does not, however, fix the particles on-mass-shell. Perhaps the requirement of one-valuedness plays an important role in quantization.

One can represent the various continuations diagrammatically. The diagrammatic rules are the following;

1. Associate with each connected 3-manifold a line with labels describing the topology of the 3-manifold and various quantum numbers of the corresponding bare state functionals.
2. The particle-number-changing transitions have as the basic vertex the three-particle vertex, and the vertex is described by the matrix G.
3. The vertices changing 3-manifold topology but preserving connectedness are described by a two-particle vertex described by the matrix G.

In this manner one can associate a diagrammatic representation with each continuation via intermediate topologies.

One-valuedness conditions state that all diagrams having the same initial and final states are equivalent, so that any reaction can be described by a unique minimal diagram.

The diagrammatic representation for the $2 \to 2$ reaction reveals that the one-valuedness conditions are analogous to the duality conditions familiar from the dual string models (Jacob, 1974; Chew and Rosenzweig, 1978), stating that the sum over the resonances in the s-channel is equivalent to the sum over the exchanges in the t-channel.

It should be emphasized, however, that the one-valuedness conditions imply restrictions on the transition amplitudes only when both s- and t-channel reactions can proceed on-shell. The assumption about crossing symmetry for $2 \to 2$-channel reactions might imply the duality conditions in their full strength.

6.4. Construction of S-Matrix

Since the relationship between bare and stationary states resembles the relationship between incoming and outgoing states in field theories, it seems natural to define the S-matrix as the matrix relating these two sets of states to each other.

Whenever possible we shall use the short-hand notations $|m)$ and $|m_s)$ for the bare and stationary states, respectively. The bare states are assumed to be orthonormalized with respect to the scalar product, whose existence follows from the phase symmetry of the action. The scalar product is assumed to be positive definite. We have

$$(m, n) = \delta(m, n) \tag{95}$$

The stationary states are not expected to be orthogonal as such and the scalar products between stationary states can be represented in the form of a matrix

$$(m_s, n_s) = (Id + G + G^+ + G^+ G)_{mn} \tag{96a}$$

Here we have used the following notations:

$$ID = \sum_t Id(t) \tag{96b}$$

$$G = \sum_{ti,\, tj} G(ti, tj) \tag{96c}$$

$$G^+ = \sum_{ti,\, tj} Gt(ti, tj) \tag{96d}$$

Since the matrix formed by the scalar products is Hermitian, it is possible to perform a unitary transformation U making this matrix diagonal. It is clear that the diagonalizing transformation mixes stationary states corresponding to different topologies. We assume, however, that the mixing is so small that there exist a natural correspondence between the bare states $|m)$ and the new diagonalized states $|\tilde{m}_s)$. The diagonalizing is necessary in order to define positive-definite transition probabilities.

In the orthogonalized basis the matrix formed by the scalar products has the form

$$(\tilde{m}_s, \tilde{n}_s) = Z(m)\, \delta_{m,n} \tag{97}$$

Here the constants $Z(m)$ are analogous to the wave function renormalization constants of the ordinary quantum field theories.

With these preliminaries we are ready to define the S-matrix and its dual via the following formula:

$$S = (n, \tilde{m}_s)/[Z(m)]^{1/2} \tag{98}$$

The unitarity of the S-matrix and thus the existence of positive-definite transition probabilities follow from the assumption that the scalar products between the bare states are positive definite.

In the orthogonalized basis the representation of the S-matrix in terms of the matrix G is given by the formula

$$S_{mn} = A_{nm}/[Z(m)]^{1/2}, \qquad A = (Id + G)U \tag{99}$$

It is rather easy to demonstrate that the nontriviality of the matrix U is necessary in order to obtain a physically acceptable S-matrix. Assuming that the U-matrix is the identity matrix, the S-matrix elements between state functionals $S^m(t)$ and $S^n(t)$ are diagonal. Thus, the S-matrix would be nontrivial only for topology-changing transitions. For instance, for the scattering of two charged particles the S-matrix would be trivial.

The mixing of different 3-topologies caused by the matrix U is necessary in order to explain Cabibbo mixing, if different fermion families correspond to different boundary component topologies. Cabibbo mixing can be identified as a mixing of different boundary topologies caused by the diagonalizing matrix U.

6.5. Evaluation of the Couplings

The evaluation of the various coupling constants is not attempted in this work. Here I consider only the general features of the problem.

A natural definition for the coupling constants is as overlap integrals of the superfield basis associated with initial and final topologies. The integral is taken over the space $\mathrm{Map}(X_{12}, H)$, where X_{12} is a singular 3-manifold intermediate between the initial and final topologies.

It is clear that the calculation reduces to that of calculating two-particle (mixing of different one-particle topologies) and three-particle vertices. It is equally clear that the direct calculation of the coupling constants is out of the question. The formal expressions derived can be used to give an insight into the general structure of the S-matrix only. Here, however, the group invariance saves the situation. The tensor product of the initial state and the conjugate of the final states formed using the coupling constant as weights of various states must be a G-singlet.

The symmetry group is indeed enormous and thus the coupling constants might be calculable, apart from a finite number of parameters, by purely group-theoretic considerations; one constructs G-singlets from the tensor product of given representations of G. Probably the construction of the vertex operators in the string model (Goddard-Olive, 1983) closely corresponds to this. The vertex operators should be interpretable as coefficients defining the singlet formed as the tensor product of three representations of the Kac–Moody algebra.

Even the arbitrary parameters appearing in the group-theoretically evaluated overlap integrals might be evaluated using the fact that the state functional is single-valued. Thus, the hopes of solving the theory are rather high.

In the pointlike limit the overlap integrals of free particle superfields over the configuration space and over theta parameters should describe various three-particle vertices. Thus we are in the situation where we can evaluate all couplings apart from an overall scaling factor at the pointlike limit once we have an orthonormalized basis of elementary particle solutions of the super-d'Alembertian.

It is rather easy to see that the expected selection rule and general coupling structure hold for the couplings of spin-one bosons to fermions. This is due to the appearance of the charge matrices in the definition of the bosonic superfields. The current obtained after the integration over the theta parameters and CP_2 degrees of freedom has the structure of an interaction term of an M^4 field theory.

The general properties of the gravitational interaction follow from the general form of the solution of the super-d'Alembertian in a very beautiful manner. The graviton contains two pairs of theta parameters and this implies that the two-fermion-graviton vertex is vanishing in the lowest order of the superfield expansion of each particle. The second term in the expansion of the particle is of the form

$$\bar{\theta}\nabla\theta T \equiv XT \tag{100}$$

Only the M^4 part of X contributes to the gravitational coupling. Thus, the gravitational coupling is proportional to the 4-momentum of the particle. Furthermore, the weakness of the coupling is also easy to understand. The expansion for the solution is of the form considered only because we have taken as the length unit the Planck mass. In conventional units the term would be proportional to a parameter R equal to the CP_2 radius.

How can one then understand the strength of the gluonic coupling? The point is that only the CP_2 part of the operator ∇ is involved in the calculation of the gluonic coupling. Since the gradient of the colored states is of the order of magnitude given by the inverse of the Planck length, gluonic coupling is indeed of the required order of magnitude, in complete accordance with the concept of the color gravitational analogy.

7. SUMMARY

In this paper a new technical realization of the quantization program described in the introduction has been attempted.

1. Instead of working in the space Map(X, H)/Diff, the space Map(X, H) is chosen as the basic geometric object. The general parametrization invariance is realized by assuming that superfields are Diff-invariant

fields in the space Map(X, H). The space of maps is endowed with an almost unique Diff-invariant and G[local $M^4 \times SU(3)$!]-invariant geometry.[4]

The metric of Map(X, H) turns out to be the Kähler metric and the construction of the geometry also leads to an understanding of the central extensions of the G Lie algebra. The central extensions correspond to the addition of a suitable multiple of the Kähler potential to the covariant derivative, and thus one obtains a purely geometric interpretation for the so-called "second quantization."

The centrally extended Lie algebra has the same general form as the Kac–Moody algebra appearing in the string model. The induced Kähler form defines a magnetic structure in any 3-surface. In dimension 3 this structure is exceptionally rich topologically.

2. The definition of the superfield concept is accomplished by introducing the spinor Grassmann algebra as an algebra generated by the "theta parameters" in one-to-one correspondence with a spinor basis of Map(X, H). The superfield is defined as a field having values in this Grassmann algebra. The super-d'Alembertian is defined as an operator analogous to the ordinary d'Alembertian.

The formalism leads to surprisingly strong results.

(a) When the gamma matrices appearing in the super-d'Alembertian are replaced by suitable modified gamma matrices the field equations reduce [by the constant-curvature property of Map(X, H)] to a simple algebraic equation satisfied by the field generating the solution of the super-d'Alembertian.

(b) As far as the properties of the scalar product are concerned, the theory is found to be finite. This results from a cancellation of potential divergences and is caused by the specific properties of the solution ansatz. The calculation of the scalar product leads to a formalism similar to that used to calculate transition amplitudes in free field theory.

(c) The requirement of a nontrivial scalar product implies matter-antimatter asymmetry at the level of the field equations, separate conservation of the baryon and lepton numbers at the level of the field equations, and spontaneous breaking of this symmetry. At the pointlike like limit $N = 1$ supersymmetry is implied by the same requirement.

(d) Since the isometries of Map(X, H) form the local gauge group $M^4 \times SU(3)$, the classification of the solution spectrum reduces to a purely group-theoretic problem. Classify the Diff-invariant, unitary representations of the various central extensions of the Kac–Moody algebra associated with

[4]It is probable that the local M^4 symmetry in fact extends to local Poincaré symmetry. This is in accordance with the idea that local Poincaré group is the gauge group of gravitational interactions.

G. The resulting formalism resembles closely that encountered in the quantization of string models.

(e) The pointlike limit of the theory has $N=1$ supersymmetry. The concept of R-invariance related to $N=1$ supersymmetry turns out to be very useful in the classification of the physical states.

The concept of central extension is found to give a nice realization for the idea that quarks correspond to pseudo triplet partial waves of CP_2 with anomalous hypercharge proportional to the electromagnetic charge. Different chiralities of H-spinors are found to correspond naturally to quarks and leptons.

If one defines elementary particles roughly as generators of the algebra to which physical states belong as a subset, one can identify lowest generation fermions (recall the topological explanation of the family replication phenomenon), electroweak gauge bosons, gluons, and gravitons and their super partners from the spectrum. A great number of other particles are predicted to exist.

(f) In the construction of the S-matrix the concepts of bare and stationary states play central roles. Bare states are superfields restricted to the subset of the configuration space corresponding to a given 3-manifold topology. Stationary states are obtained by continuing the bare state functionals to state functionals defined in the whole configuration space.

The continuity conditions making it possible to continue a bare state functional to whole configuration space can be solved formally and the conditions guaranteeing the uniqueness of the continuation process turn out to be analogous to the conditions defining the duality concept in the context of dual models.

The S-matrix can be defined as the matrix relating the bare and the stationary states to each other. The S-matrix is unitary provided the scalar product associated with the super-d'Alembert equation is positive definite. The calculation of the S-matrix elements reduces to the solution of the continuity conditions. Since the basis of the stationary states is not normalized to unity, quantities analogous to the wave function renormalization constants appear in the expressions for the transition probabilities.

The local $M^4 \times SU(3)$ invariance combined with the one-valuedness conditions is expected to reduce the practical evaluation of the S-matrix to a group-theoretic problem. It is found that the general form of the solution ansatz makes it possible to understand the general features of the interaction vertices: for example, the weakness of the gravitational interaction.

Summarizing, the use of the space Map(X, H) endowed with G- and diffeoinvariant geometry has made possible the formulation of a calculable theory. In particular, the concept of central extension has turned out to be especially useful. In fact, there are strong reasons to believe that the characterization of a many-particle system through the topological properties of the induced Kähler field might provide a new powerful tool for the description and understanding of the behavior of many-particle systems. Even an explanation for the dimensionality of space has emerged; in dimension 3 central extension turns out to be exceptionally rich topologically.

APPENDIX: MODIFIED GAMMA MATRICES AND SPINOR COHOMOLOGY

The Kähler form of CP_2 is covariantly constant and its square gives the negative of the metric tensor. Thus the modified gamma matrices defined by

$$\Gamma^k_\pm = (h^k_l + iJ^k_l)\Gamma^l/2 \tag{A1}$$

are covariantly constant quantities and define complexification of the gamma matrix algebra of CP_2.

The modified gamma matrices obey an algebra isomorphic to that obeyed by fermionic creation and annihilation operators,

$$\{\Gamma^k_+, \Gamma^l_-\} = h^{kl} Id; \qquad \{\Gamma^k_\pm, \Gamma^l_\pm\} = 0 \tag{A2}$$

as is seen by multiplying the gamma matrices with factors $i^{1/2}$.

The definition of the modified Dirac operator is obvious; one needs only to replace the gamma matrices appearing in these operators with the annihilation-type gamma matrices. The square of the modified Dirac operator of CP_2 vanishes identically.

One can extend the definition of the modified gamma matrices to the case of the space Map(X, H) because of its Kähler structure. Since the modified gamma matrices and their Hermitian conjugates obey the algebra of fermionic creation and annihilation operators, one can require in the infinite-dimensional case that spinors are obtained from some "vacuum spinor" by applying a finite number of creation operator-like-modified gamma matrices.

In the infinite-dimensional case the appearance of only annihilation-type gamma matrices in the modified Dirac operator has an important consequence; only a finite number of them give a nonvanishing result when applied to a configuration space spinor. Concerning the calculability of the theory, the modification of the Dirac operator appearing in the field equations thus seems highly desirable.

The modified Dirac operator operator of CP_2, denote it by $d_\pm$, defines "spinor cohomology" in H. One can define closed ($du = 0$), exact ($u = dv$), and cohomologically nontrivial (closed but nonexact) spinors. Furthermore, one can define the cohomology group as the linear space of the cohomologically nontrivial spinors.

In order to understand the properties of the spinor cohomology it is advantageous to use complex coordinates $(\xi^1, \xi^2, \bar{\xi}^1, \bar{\xi}^2)$ for CP_2 (Pitkänen, 1981, 1983, 1985, 1986; Eguchi et al., 1980). In these coordinates the operator d_+ has a surprisingly simple form,

$$d_+ = \Gamma^{\bar{k}} D_{\bar{k}} \tag{A3}$$

Thus, d_+ is simply proportional to the half of the Dirac operator that acts on the variables $\bar{\xi}^k$.

The square of $d_\pm$ vanishes, since the curvature form of the spinor connection satisfied has no components of type F_{kl} or $F_{\bar{k}\bar{l}}$.

The vacuum spinor annihilated by the annihilation operator-type gamma matrices is a closed spinor. In the case of CP_2 this spinor has quantum numbers of a right-handed neutrino or a right-handed electron, depending on which gammas are interpreted as annihilation-type gamma matrices. The right-handed neutrino is a closed spinor because of its covariant constancy.

The spinors satisfying the condition

$$D_{\bar{k}} u = 0 \tag{A4}$$

are closed. These conditions can be regarded as a generalization of the analyticity conditions obtained by replacing ordinary derivatives with covariant derivatives. The integrability conditions associated with these equations are satisfied identically. The existence of analytic noncovariantly constant spinors is, however, improbable (only the constant analytic function in the complex plane is everywhere regular).

The concept of spinor cohomology also generalizes to the case of Map(X, H). Now the spinor annihilated by all modified gamma matrices (but not M^4 gamma matrices) is closed in the spinor cohomology and corresponds to the "vacuum spinor."

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REFERENCES

Berezin, F. A. (1966). *Method of Second Quantization*, Academic Press, New York.

Chew, G., and Rosenzweig, C. (1978). *Physics Reports*, **41**C.

Cronin, J. W. (1981). *Review of Modern Physics*, **53**.

De la Harpe, P. (1972). *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space*, Lecture Notes in Mathematics, No. 285, Springer, New York.

Eguchi, T., Gilkey, B., and Hanson, J. ((1980). *Physics Reports*, **66**, 6.

Fitch, V. L. (1981). *Review of Modern Physics*, **53**.

Freed, D. S. (1985). The geometry of loop spaces, Thesis, University of California, Berkeley.

Gibbons, G. W., and Pope, C. N. (1978). *Communications in Mathematical Physics*, **61**, 239 (1978).

Goddard, P., and Olive, D. (1984). *Vertex Operators in Mathematics and Physics*, Springer, New York.

Goldstein, D. L. (1975). *States of Matter*, Prentice-Hall, New Jersey.

Goodman, R., and Wallach, N. (1985). *Jahrbuch für Mathematik*, **347**, 69.

Hawking, S. W., and Pope, C. N. (1978). *Physics Letters*, **73B**, 42.

Helgason, S. (1962). *Differential Geometry and Symmetric Spaces*, Academic Press, New York.

Jacob, H. (1974). *Dual Theory*, North-Holland, Amsterdam.

Jehle, H. (1977). *Physical Review*, **D15**.

Kac, V. G. (1983). *Infinite Dimensional Lie Algebras*, Birkhauser, Boston.

Pitkänen, M. (1981). *International Journal of Theoretical Physics*, **20**, 843.

Pitkänen, M. (1983). *International Journal of Theoretical Physics*, **22**, 575.

Pitkänen, M. (1985). *International Journal of Theoretical Physics*, **24**, 775.

Pitkänen, M. (1986). *International Journal of Theoretical Physics*, **25**(1).

Pope, C. N. (1980). *Physics Letters*, **97B**, 3, 4.

Pope, C. N. (1982). Preprint ICTP/81/82-17.

Rolfsen, D. (1976). *Knots and Links*, Publish or Perish, Berkeley, California.

Schwartz, J. H. (1985). Caltech Preprint, CALT-68-1252.

Seifert, H., and Threlfall, W. (1931). *Mathematische Annalen*, **104**, 1.

Seifert, H., and Threlfall, W. (1932). *Mathematische Annalen*, **107**, 543.

Seifert, H., and Threlfall, W. (1950). *Canadian Journal of Mathematics*, **2**, 1.

Stelle, K. S. (1983). In *Gauge Theories of the Eighties*, R. Raitio and J. Lindfors, eds., Lecture Notes in Physics, Springer, New York.

Stinespring, W. F. (1965). *Journal of Mathematics and Mechanics*, **14**, 315-322.

Volkov, D., and Akulov, V. (1973). *Physics Letters*, **46B**.

Wess, J., and Zumino, B. (1974). *Nuclear Physics B*, **70**.